Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives.**
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

Do not write in the table to the right.

Problem	Points	Score
1	16	
2	16	
3	17	
4	16	
5	16	
6	19	
Total:	100	

1. 16 Points Find the *length* of the curve $y = \int_2^x \sqrt{3t^4 - 1}\, dt$ from $x = -2$ to $x = -1$.

Solution: By Fundamental Theorem of Calculus Part I, we have $\frac{dy}{dx} = \frac{d}{dx}\left(\int_2^x \sqrt{3t^4-1}\, dt\right) = \sqrt{3x^4-1}$. Thus for $x \in [-2, -1]$, we have

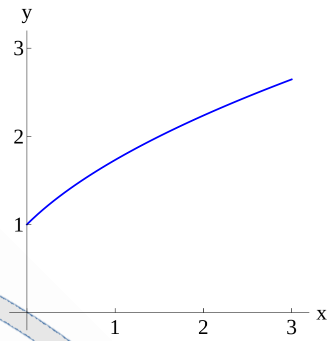
$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\sqrt{3x^4-1}\right)^2} = \sqrt{3x^4} = \sqrt{3}x^2.$$

Hence the length we want is then

$$L_{-2}^{-1} = \int_{-2}^{-1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{-2}^{-1} \sqrt{3}x^2\, dx = \sqrt{3}\left[\frac{x^3}{3}\right]_{-2}^{-1} = \frac{\sqrt{3}}{3}\left[-1-(-2)^3\right] = \frac{\sqrt{3}}{3}(-1+8) = \boxed{\frac{7\sqrt{3}}{3}}$$

p.191, pr.54

2. [16 Points] Find the *area of the surface* generated by revolving the curve about the x-axis. $y = \sqrt{2x+1},\ 0 \le x \le 3.$



Solution: $\dfrac{dy}{dx} = \dfrac{d}{dx}(\sqrt{2x+1}) = \dfrac{1}{\sqrt{2x+1}} \Rightarrow \sqrt{1+\left(\dfrac{dy}{dx}\right)^2} = \sqrt{1+\left(\dfrac{1}{\sqrt{2x+1}}\right)^2} = \sqrt{\dfrac{2x+2}{2x+1}}.$

Now $y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2} = \sqrt{2x+1}\sqrt{\dfrac{2x+2}{2x+1}} = \sqrt{2x+2}$. Hence the area of the surface of revolution is

$$\begin{aligned} S &= 2\pi\int_0^3 y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\sqrt{2}\pi\int_0^3 \sqrt{x+1}\,dx \\ &= 2\sqrt{2}\pi\left[\frac{2}{3}(x+1)^{3/2}\right]_0^3 = 2\sqrt{2}\pi\frac{2}{3}(8-1) = \boxed{\frac{28\pi\sqrt{2}}{3}} \end{aligned}$$

p.241, pr.45

3. [17 Points] Use only **optimization** to find the point on the line $\dfrac{x}{a}+\dfrac{y}{b}=1$ that is *closest* to the origin.

Solution: Suppose $P(x,y)$ is the point on this line that is closest to the origin. Let $d = \sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2}$ and $\dfrac{x}{a}+\dfrac{y}{b}=1 \Rightarrow y = -\dfrac{b}{a}x+b.$

We can minimize d by minimizing $D = \left(\sqrt{x^2+y^2}\right)^2 = x^2+\left(-\dfrac{b}{a}x+b\right)^2 \Rightarrow D' = 2x+2\left(-\dfrac{b}{a}x+b\right)\left(-\dfrac{b}{a}\right) = 2x+\dfrac{2b^2}{a^2}x-\dfrac{2b^2}{a}$. Hence $D' = 0 \Rightarrow 2\left(x+\dfrac{b^2}{a^2}x-\dfrac{b^2}{a}\right) = 0 \Rightarrow x = \dfrac{ab^2}{a^2+b^2}$ is the critical point $\Rightarrow y = -\dfrac{b}{a}\left(\dfrac{ab^2}{a^2+b^2}\right)+b = \dfrac{a^2b}{a^2+b^2}$. Thus $D'' = 2+\dfrac{2b^2}{a^2} \Rightarrow D''\left(\dfrac{ab^2}{a^2+b^2}\right) = 2+\dfrac{2b^2}{a^2} > 0 \Rightarrow$ the critical point is local minimum by the Second Derivative Test, $\Rightarrow \left(\dfrac{ab^2}{a^2+b^2},\dfrac{a^2b}{a^2+b^2}\right)$ is the point on the line $\dfrac{x}{a}+\dfrac{y}{b}=1$ that is closest to the origin.

p.222, pr.28

4. Given the curve $y = x^{2/3}(x-5)$, $\quad y' = \dfrac{5}{3}x^{-1/3}(x-2)$, $\quad y'' = \dfrac{10}{9}x^{-4/3}(x+1)$.

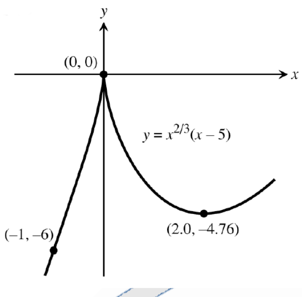
(a) [8 Points] Find the open intervals where the graph is *increasing* and *decreasing*. Find the *local maximum* and *minimum* values.

Solution: Notice that $y' > 0$ if and only if $x < 0$ or $x > 2$. Thus the curve is rising on $(-\infty,0)$ and $(2,\infty)$, and falling on $(0,2)$. There is a local minimum at $x = 2$ and a local maximum at $x = 0$.

p.241, pr.45

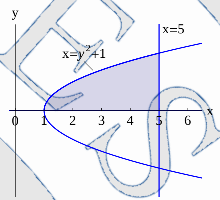
(b) [8 Points] Determine where the graph is *concave up* and *concave down*, and find any *inflection points*.

Solution: Notice that $y'' > 0$ if and only if $-1 < x < 0$ or $x > 0$. Therefore the curve is concave up on $(-1,0)$ and $(0,\infty)$, and concave down on $(-\infty,-1)$. The point of inflection is $(-1,-4)$ and a cusp at $(0,0)$. The graph is as follows.



p.241, pr.45

5. 16 Points Find the *volume* of the solid generated by revolving the region about the x-axis bounded by
$x = y^2 + 1$, $x = 5$, $y = 0$, and $y \geq 0$.



Solution: If $0 \leq y \leq 2$, then a horizontal strip of the given region "at" y has length $5-(y^2+1)$ and moves around a circle of radius y, so the volume generated rotation of that region around x-axis is

$$\begin{aligned} V &= \int_c^d 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dy = \int_0^2 2\pi y(5-y^2-1)\,dy \\ &= 2\pi\int_0^2 (4y-y^3)\,dy \\ &= 2\pi\left[2y^2-\frac{1}{4}y^4\right]_0^2 = 2\pi(8-4) = 8\pi. \end{aligned}$$

Alternatively, when the method of disks is used, one gets

$$\begin{aligned} V &= \int_1^5 \pi\,[R(x)]^2\,dx = \pi\int_1^5\left[\sqrt{x-1}\right]^2 dx \\ &= \pi\int_1^5 (x-1)\,dx \\ &= \pi\left[\frac{1}{2}x^2 - x\right]_1^5 \\ &= \pi\left[\left(\frac{25}{2}-5\right)-\left(\frac{1}{2}-1\right)\right] \\ &= \boxed{8\pi} \end{aligned}$$

p.236, pr.27

6. (a) 9 Points $\displaystyle\int_0^{\pi/6}(1-\cos(3t))\sin(3t)\,dt = ?$

Solution: Let $u = 1-\cos(3t)$. Then $du = 3\sin(3t)\,dt$. When $t=0$, we have $u = 1-\cos(0) = 0$ and when $t = \frac{\pi}{6}$, we have $u = 1-\cos(\pi/2) = 1$. Hence, we have

$$\int_0^{\pi/6}(1-\cos(3t))\sin(3t)\,dt = \frac{1}{3}\int_0^{\pi/6}\underbrace{(1-\cos(3t))}_{1-u}\underbrace{3\sin(3t)\,dt}_{du} = \int_0^1 \frac{1}{3}u\,du = \left[\frac{1}{3}\left(\frac{u^2}{2}\right)\right]_0^1 = \frac{1}{6}(1)^2 - \frac{1}{6}(0)^2 = \boxed{\frac{1}{6}}$$

p.297, pr.11(a)

(b) 10 Points $\displaystyle\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx = ?$

Solution: Let $y = 1+\sqrt{x}$. Then $dy = \dfrac{1}{2\sqrt{x}}\,dx$. Therefore, we have

$$\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx = 2\int \underbrace{\frac{1}{(1+\sqrt{x})^2}}_{\frac{1}{y^2}} \underbrace{\frac{1}{2\sqrt{x}}\,dx}_{dy} = 2\int \frac{1}{y^2}\,dy = 2\left(-\frac{1}{y}\right) + C = \boxed{-\frac{2}{1+\sqrt{x}} + C}$$

p.290, pr.6