Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

Problem	Points	Score
1	15	
2	23	
3	25	
4	20	
5	17	
Total:	100	

Do not write in the table to the right.

1. 15 Points Find the *area of the surface* generated by revolving $x = 2\sqrt{4-y}$, $0 \le y \le 15/4$ about y-axis.

Solution: The surface area formula we shall use is $S = \int_c^d 2\pi x\sqrt{1+(dx/dy)^2}\,dy$

$$x = 2\sqrt{4-y} \Rightarrow \frac{dx}{dy} = -\frac{1}{\sqrt{4-y}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{4-y} \Rightarrow S = \int_0^{15/4} 2\pi(2\sqrt{4-y})\sqrt{1+\frac{1}{4-y}}\,dy$$

$$S = 4\pi\int_0^{15/4} \cancel{\sqrt{4-y}}\,\frac{\sqrt{5-y}}{\cancel{\sqrt{4-y}}}\,dy = 4\pi\int_0^{15/4}\sqrt{5-y}\,dy \qquad \boxed{u = 5-y, \qquad du = -dy \Rightarrow}$$

$$= -4\pi\int_5^{5/4}\sqrt{u}\,du = -4\pi\left[\frac{u^{3/2}}{3/2}\right]_5^{5/4} = -\frac{8\pi}{3}\left[\left(\frac{5}{4}\right)^{3/2} - (5)^{3/2}\right]$$

$$= -\frac{8\pi}{3}5^{3/2}\left[\left(\frac{1}{4}\right)^{3/2} - 1\right] = -\frac{8\pi}{3}5\sqrt{5}\left(\frac{1}{8}-1\right)$$

$$\longrightarrow S = -\frac{8\pi}{3}5\sqrt{5}\left(-\frac{7}{8}\right) = \boxed{\frac{35\pi\sqrt{5}}{3}}$$

p.336, pr.19

2. (a) 10 Points Find dy/dx if $y = \left(\frac{\sqrt{x}}{1+x}\right)^2$.

Solution: By the chain rule, we have

$$\frac{dy}{dx} = 2\left(\frac{\sqrt{x}}{1+x}\right)\frac{\frac{1}{2\sqrt{x}}(1+x)-\sqrt{x}(1)}{(1+x)^2} = \frac{dy}{dx} = \frac{2\sqrt{x}}{1+x}\cdot\frac{1-x}{2\sqrt{x}(1+x)^2} = \boxed{\frac{1-x}{(1+x)^3}}$$

p.176, pr.31

(b) 13 Points For what values of a and b will

$$f(x) = \begin{cases} ax+b, & x \le -1 \\ ax^3+x+2b, & x > -1 \end{cases}$$

be *differentiable at every x*?

Solution: The function f is clearly differentiable except possibly at $x=-1$. Suppose f is also differentiable at $x=-1$. Then it must be continuous there. Hence we must have $\lim_{x\to-1^+} f(x) = \lim_{x\to-1^+} f(x) \Rightarrow -a+b = -a-1+2b \Rightarrow b=1$. But here we have

$$\begin{aligned} f'_-(-1) &= \lim_{h\to0^-}\frac{f(-1+h)-f(-1)}{h} \\ &= \lim_{h\to0^-}\frac{(a(-1+h)+b)-(a(-1)+b)}{h} = \lim_{h\to0^-}\frac{\cancel{-a}+ah\cancel{+b}\cancel{+a}-\cancel{b}}{h} \\ &= \lim_{h\to0^-}\frac{a\cancel{h}}{\cancel{h}} == \lim_{h\to0^-}(a) = \boxed{a} \end{aligned}$$

and

$$\begin{aligned} f'_+(-1) &= \lim_{h\to0^+}\frac{f(-1+h)-f(-1)}{h} \\ &= \lim_{h\to0^+}\frac{(a(-1+h)^3+(-1+h)+2b)-(a(-1)^3+(-1)+2b)}{h} \\ &= \lim_{h\to0^+}\frac{ah^3-3ah^2+3ah-\cancel{a}-\cancel{1}+h+\cancel{2b}+\cancel{a}+\cancel{1}-\cancel{2b}}{h} \\ &= \lim_{h\to0^+}\frac{\cancel{h}(ah^2-3ah+3a+1)}{\cancel{h}} == \lim_{h\to0^-}(ah^2-3ah+3a+1) = \boxed{3a+1} \end{aligned}$$

Therefore, f is differentiable at $x=-1$ iff $f'_-(-1) = f'_+(-1)$ iff $a = 3a+1$, that is iff $\boxed{a=-1/2}$. It then easily follows that the unique function

$$f(x) = \begin{cases} -\frac{1}{2}x+1, & x \le -1 \\ -\frac{1}{2}x^3+x+2, & x > -1 \end{cases}$$

is *differentiable at every x* and has the desired property.

p.182, pr.18

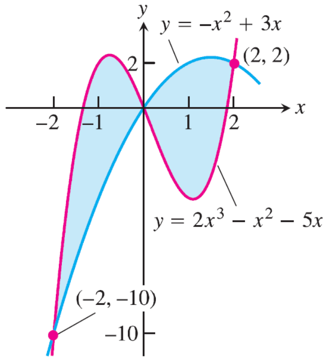
3. (a) 10 Points $\displaystyle\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\,\mathrm{d}x = ?$

Solution: Let $u = 1+\sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}$, and so

$$\begin{aligned} \int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\,\mathrm{d}x &= 2\int \underbrace{(1+\sqrt{x})^{1/3}}_{u^{1/3}}\underbrace{\frac{1}{2\sqrt{x}}\,\mathrm{d}x}_{du} = 2\int u^{1/3}\,\mathrm{d}u = 2\left[\frac{u^{4/3}}{4/3}\right]+C \\ &= \frac{6}{4}u^{4/3}+C = \boxed{\frac{3}{2}(1+\sqrt{x})^{4/3}+C} \end{aligned}$$

p.302, pr.53

(b) 15 Points Find the total area of the shaded region.



Solution: The figure shows that the given curves meet at three points $(-2,-10)$, $(0,0)$, and $(2,2)$. The area of region on the left is

$$A_1 = \int_{-2}^{0} [\underbrace{2x^3 - x^2 - 5x}_{\text{upper curve}} - (\underbrace{-x^2+3x}_{\text{lower curve}})]\,dx = \int_{-2}^{0} (2x^3 - 8x)\,dx$$

$$= \left[\frac{2x^4}{4} - \frac{8x^2}{2}\right]_{-2}^{0} = 0 - (8-16) = \boxed{8}$$

The area of region on the right is

$$A_1 = \int_{0}^{2} [\underbrace{-x^2+3x}_{\text{upper curve}} - (\underbrace{2x^3 - x^2 - 5x}_{\text{lower curve}})]\,dx = \int_{0}^{2} (8x - 2x^3)\,dx$$

$$= \left[\frac{8x^2}{2} - \frac{2x^4}{4}\right]_{0}^{2} = (16-8) = \boxed{8}$$

Therefore the total area of the two regions is $A = A_1 + A_2 = 8 + 8 = \boxed{16}$

p.302, pr.38

4. (a) 10 Points $\lim\limits_{x\to 1} \dfrac{1-\sqrt{x}}{1-x}$ =? (**Do not use L'Hôpital's Rule**)

Solution:

$$\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} = \lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}} = \lim_{x\to 1} \frac{\cancel{1-x}}{(\cancel{1-x})(1+\sqrt{x})} = \lim_{x\to 1} \frac{1}{1+\sqrt{x}} = \frac{1}{1+\sqrt{1}} = \boxed{\frac{1}{2}}$$

p.97, pr.11

(b) 10 Points $\lim\limits_{x\to\infty} \dfrac{x^{2/3} + x^{-1}}{x^{2/3} + \cos^2 x}$ =? (**Do not use L'Hôpital's Rule**)

Solution:

$$\lim_{x\to\infty} \frac{x^{2/3}+x^{-1}}{x^{2/3}+\cos^2 x} = \lim_{x\to\infty} \frac{x^{2/3}+x^{-1}}{x^{2/3}+\cos^2 x} \cdot \frac{1/x^{2/3}}{1/x^{2/3}} = \lim_{x\to\infty} \frac{1+(1/x^{5/3})}{1+(\frac{\cos^2 x}{x^{2/3}})} = \frac{1+\lim\limits_{x\to\infty}(1/x^{5/3})}{1+\lim\limits_{x\to\infty}(\frac{\cos^2 x}{x^{2/3}})}$$

Now the first of last two limits is $\lim\limits_{x\to\infty}(1/x^{5/3}) = 0$ and the other also equals zero by Sandwich Theorem, as

$$0 \le \cos^2 x \le 1 \Rightarrow 0 < \frac{\cos^2 x}{x^{2/3}} \le \frac{1}{x^{2/3}}, \quad \forall x \ne 0 \Rightarrow 0 < \lim_{x\to\infty}\left(\frac{\cos^2 x}{x^{2/3}}\right) \le \lim_{x\to\infty} \frac{1}{x^{2/3}} = 0 \Rrightarrow \lim_{x\to\infty}\left(\frac{\cos^2 x}{x^{2/3}}\right) = 0.$$

Hence we get

$$\lim_{x\to\infty} \frac{x^{2/3}+x^{-1}}{x^{2/3}+\cos^2 x} = \frac{1+\lim\limits_{x\to\infty}(1/x^{5/3})}{1+\lim\limits_{x\to\infty}(\frac{\cos^2 x}{x^{2/3}})} = \frac{1+0}{1+0} = \boxed{1}$$

p.97, pr.24

5. 17 Points Let $y = ax^3 + bx^2 + cx$. Find the values of contants a, b, and c so that its graph has a *local minimum* at $x=-1$, a *local maximum* at $x=3$ **and** a *point of inflection* at $(-1,-2)$. *Justify your answer.*

Solution: Assume that $y = ax^3 + bx^2 + cx$ has the properties we want. Then $\dfrac{dy}{dx} = 3ax^2 + 2bx + c$ and $\dfrac{d^2y}{dx^2} = 6ax + 2b$. Since the graph has a local minimum at $x=-1$, we must have $\left.\dfrac{dy}{dx}\right|_{x=-1} = 0$. Therefore $3a - 2b + c = 0$. Now since the graph has a local maximum at $x = 3$, we must have $\left.\dfrac{dy}{dx}\right|_{x=3} = 0$ and so $27a + 6b + c = 0$. Moreover, it is known that the graph has a point of inflection at $(1,11)$. This yields $\left.\dfrac{d^2y}{dx^2}\right|_{x=1} = 0$ and $y(1) = 11$. This gives $6a(1) + 2b = 0 \Rightarrow 6a + 2b = 0$ and

$a(1)^3 + b(1)^2 + c(1) = 11 \Rightarrow a+b+c = 11$. We now solve the system

$$\left.\begin{array}{l} 27a+6b+c=0 \\ 3a-2b+c=0 \\ a+b+c=11 \\ 6a+2b=0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 27a+6b+c=0 \\ 3a-2b+c=0 \\ a+(-3a)+c=11 \\ b=-3a \end{array}\right\} \Rightarrow \left.\begin{array}{l} 27a+6b+c=0 \\ 3a-2(-3a)+(11+2a)=0 \\ c=11+2a \end{array}\right\} \Rightarrow \left.\begin{array}{l} 27a+6b+c=0 \\ 11a=-11 \end{array}\right\} \Rightarrow \boxed{\left.\begin{array}{l} a=-1 \\ b=3 \\ c=9 \end{array}\right\}}$$

This produces $\boxed{y = -x^3 + 3x^2 + 9x}$. Now we need to check this function fulfills the requirements. We simply apply the second derivative test. Notice that $y' = -3x^2 + 6x + 9$ and $y'' = -6x + 6$. We see that $y''(3) = -6(3) + 6 = -12 < 0$ shows that at $x = 3$ there is a local maximum and $y''(-1) = -6(-1) + 6 = 12 > 0$ shows that the graph has a local minimum at $x = -1$. Finally, since $y'' = -6x + 6$ is positive for $x < 1$ and is negative for $x > 1$, we see that the graph has a point of inflection at $(1, 11)$. Hence there is *exactly one* such function.

p.213, pr.112