Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- Calculators, cell phones off and away!.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place [a box around your answer] to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

Do not write in the table to the right.

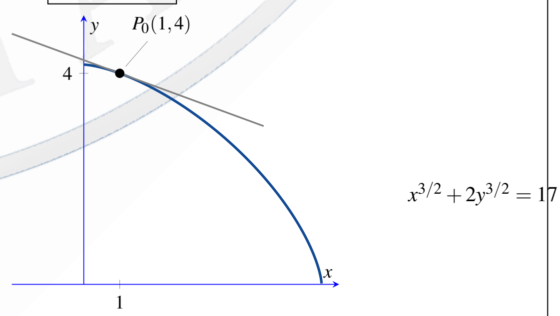
Problem	Points	Score
1	20	
2	22	
3	18	
4	20	
5	20	
Total:	100	

1. (a) [10 Points] Find the equation of the line tangent to the curve $x^{3/2}+2y^{3/2}=17$ at the point $P_0(1,4)$.

Solution: Assuming that this equation defines implicitly y as a function of x, first differentiate the given equation with respect to x to find the slope of this tangent.

$$\frac{d}{dx}\left(x^{3/2}+2y^{3/2}\right)=\frac{d}{dx}(17)$$
$$\frac{d}{dx}\left(x^{3/2}\right)+2\frac{d}{dx}\left(y^{3/2}\right)=0$$
$$\frac{3}{2}x^{1/2}+3y^{1/2}\frac{dy}{dx}=0$$
$$\frac{dy}{dx}=-\frac{\frac{3}{2}x^{1/2}}{3y^{1/2}}=-\frac{x^{1/2}}{2y^{1/2}} \Rightarrow \text{slope } = m=\left[\frac{dy}{dx}\right]_{(x,y)=(1,4)}=\left[-\frac{x^{1/2}}{2y^{1/2}}\right]_{(x,y)=(1,4)}=-\frac{(1)^{1/2}}{2(4)^{1/2}}=\boxed{-\frac{1}{4}}$$

Hence the equation for the tangent line is $y-4=-\frac{1}{4}(x-1) \Rightarrow 16-4y=x-1 \Rightarrow \boxed{x+4y=17}$.



p.423, pr.34

(b) [10 Points] For what values of c is the curve $y=\frac{c}{x+1}$ tangent to the line through the points $(0,3)$ and $(5,-2)$?

Solution: Let L be the line through $(0,3)$ and $(5,-2)$. If m is the slope for L, then $m = \frac{-2-3}{5-0} = -1$. Hence L has equation $y-3 = (-1)(x-0) \Rightarrow \boxed{y = -x+3}$. Next, we compute $\frac{d}{dx}\left(\frac{c}{x+1}\right) = -\frac{c}{(x+1)^2}$. Suppose L is tangent to curve at $x = x_0$. Then this implies

$$\left[-\frac{c}{(x+1)^2}\right]_{x=x_0} = m \Rightarrow -\frac{c}{(x_0+1)^2} = -1 \Rightarrow (x_0+1)^2 = c \Rightarrow x_0+1 = \pm\sqrt{c} \Rightarrow x_0 = -1 \pm \sqrt{c}$$

Next the point $A(x_0, \frac{c}{x_0+1})$ is on the curve. Similarly the point $B(x_0, -x_0+3)$ lies on L. Every tangent must meet the curve at $x = x_0$. Therefore $A = B$. Hence as $c \neq 0$, as otherwise the curve is $y = 0$ and cannot be tangent L, we have

$$\frac{c}{x_0+1} = -x_0+3 \Rightarrow (x_0+1)(-x_0+3) = c$$

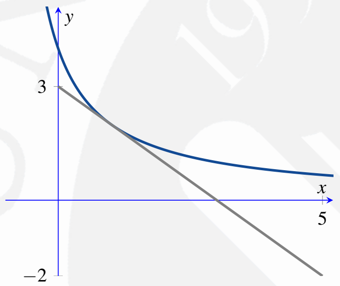
Suppose now $x_0 = -1+\sqrt{c}$. Then $x_0+1 = \sqrt{c}$ and $-x_0+3 = 4-\sqrt{c}$. Hence

$$(x_0+1)(-x_0+3) = c \Leftrightarrow \sqrt{c}(4-\sqrt{c}) = c \Leftrightarrow \cancel{\sqrt{c}}(4-\sqrt{c}) = \cancel{\sqrt{c}}\sqrt{c} \Leftrightarrow 4-\sqrt{c} = \sqrt{c} \Leftrightarrow 2\sqrt{c} = 4 \Leftrightarrow \sqrt{c} = 2 \Leftrightarrow \boxed{c = 4}$$

Similarly, suppose $x_0 = -1-\sqrt{c}$. Then $x_0+1 = -\sqrt{c}$ and $-x_0+3 = 4+\sqrt{c}$. Therefore

$$(x_0+1)(-x_0+3) = c \Leftrightarrow -\sqrt{c}(4+\sqrt{c}) = c \Leftrightarrow -\sqrt{c}(4+\sqrt{c}) = c \Leftrightarrow 4+\sqrt{c} = -\sqrt{c} \Leftrightarrow \sqrt{c} = -2$$

which is impossible. Hence the only possible value is $c = 4$.



p.387, pr.32

2. (a) [12 Points] Find the *volume* of solid generated by revolving the region bounded by $y = x^2-2x$, and $y = 0$ about the line $x = 2$.

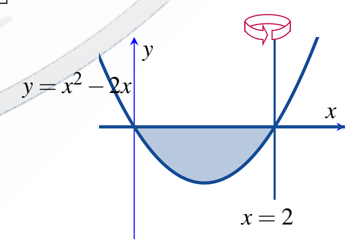
Solution: If $0 \le x \le 2$, then a vertical strip of the given region "at" x has length $-(x^2-2x)$ and moves around a circle of radius $2-x$, so the volume generated rotation of that region around $x = 2$ is

$$V = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx = 2\pi\int_0^2 (2-x)(-(x^2-2x))\,dx = 2\pi\int_0^2 (2-x)(2x-x^2)\,dx$$

$$= 2\pi\int_0^2 (4x-2x^2-2x^2+x^3)\,dx$$

$$= 2\pi\int_0^2 (x^3-4x^2+4x)\,dx$$

$$= 2\pi\left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2\right]_0^2 = 2\pi(4 - \frac{32}{3} + 8) = \frac{2\pi}{3}(36-32) = \boxed{\frac{8\pi}{3}}$$



p.345, pr.32(c)

(b) [10 Points] Find the *length* of the curve $x = y^{2/3}$ from $y = 1$ to $y = 8$.

Solution: First note that

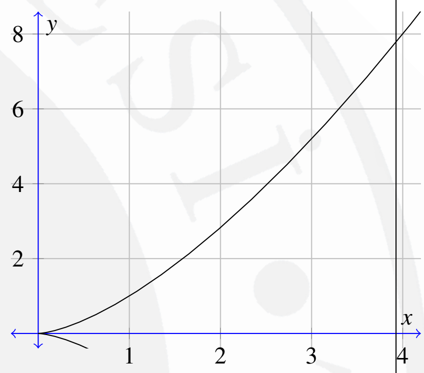
$$x = y^{2/3} \Rightarrow \frac{dx}{dy} = \frac{2}{3}y^{-1/3} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{4y^{-2/3}}{9}.$$

Therefore

$$L = \int_1^8 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,\mathrm{d}y = \int_1^8 \sqrt{1+\frac{4}{9y^{2/3}}}\,\mathrm{d}y = \int_1^8 \frac{\sqrt{9y^{2/3}+4}}{3y^{1/3}}\,\mathrm{d}y$$
$$= \frac{1}{3}\int_1^8 \sqrt{9y^{2/3}+1}\left(y^{-1/3}\right)\mathrm{d}y.$$

Now let $u = 9y^{2/3}+4$ and so $du = 6y^{-1/3}\,\mathrm{d}y$. When $y=1$, we have $u=13$ and when $y=8$, we have $u=40$. Hence the curve has length

$$L = \frac{1}{18}\int_{13}^{40} u^{1/2}\,\mathrm{d}u = \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_{13}^{40} = \frac{1}{27}\left[40^{3/2}-13^{3/2}\right] \boxed{\approx 7.634}$$



p.377, pr.24

3\. (a) 10 Points Find the value of $\displaystyle\int_1^4 \frac{\ln 2 \log_2 x}{x}\,\mathrm{d}x$.

Solution: Let $u = \log_2 x$ and so $du = \dfrac{1}{(\ln 2)x}\,\mathrm{d}x$. When $x=1$, we have $u=0$ and when $x=4$, we have $u = \log_2 4 = \log_2 2^2 = 2$. Hence

$$\int_1^4 \frac{\ln 2 \log_2 x}{x}\,\mathrm{d}x = (\ln 2)^2 \int_1^4 \log_2 x \frac{1}{x(\ln 2)}\,\mathrm{d}x$$
$$= (\ln 2)^2 \int_0^2 u\,\mathrm{d}u$$
$$= (\ln 2)^2 \left[\frac{1}{2}u^2\right]_0^2 = 2(\ln 2)^2$$

p.317, pr.14(b)

(b) 8 Points Find the limit $\displaystyle\lim_{x\to\infty} x^{1/\ln x}$.

Solution: The limit leads to the indeterminate form ∞^0. Let $f(x) = x^{1/\ln x}$. Then $\ln f(x) = \dfrac{\ln x}{\ln x} = 1$. Therefore

$$\lim_{x\to\infty} x^{1/\ln x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \mathrm{e}^{\ln f(x)} = \mathrm{e}^1 = \boxed{\mathrm{e}}$$

p.241, pr.65(a)

4\. (a) 8 Points Evaluate the integral $\displaystyle\int_1^9 \frac{(5-\sqrt{x})^{1/3}}{\sqrt{x}}\,\mathrm{d}x$.

Solution: Let $u = 5 - \sqrt{x}$ and so $du = -\dfrac{1}{2\sqrt{x}}\,\mathrm{d}x$. When $x = 1$, we have $u = 4$ and when $x = 9$, we have $u = 2$. Hence

$$\begin{aligned}\int_1^9 \frac{(5-\sqrt{x})^{1/3}}{\sqrt{x}}\,\mathrm{d}x &= -2\int_1^9 \frac{(5-\sqrt{x})^{1/3}}{-2\sqrt{x}}\,\mathrm{d}x \\ &= -2\int_4^2 u^{1/3}\,\mathrm{d}u \\ &= +2\left[\frac{u^{4/3}}{4/3}\right]_2^4 = \frac{3}{2}(4^{4/3} - 2^{4/3}) = \boxed{3(2\sqrt[3]{4} - \sqrt[3]{2})}\end{aligned}$$

p.241, pr.65(a)

(b) 12 Points Find the total area of the shaded region.

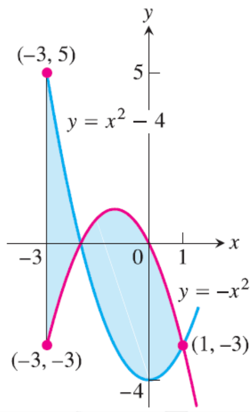
Solution: Let R denote this region.

$$R = \underbrace{\{(x,y)\in\mathbb{R}^2 \mid -x^2-2x \le y \le x^2-4, -3 \le x \le -2\}}_{R_1} \cup \underbrace{\{(x,y)\in\mathbb{R}^2 \mid x^2-4 \le y \le -x^2-2x, -2}_{R_2}$$

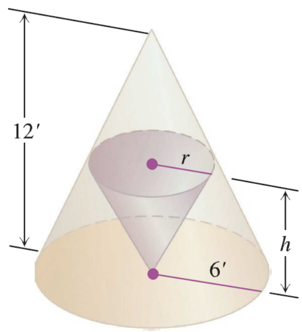
Notice that $R = R_1 \cup R_2$ and $R_1 \cap R_2 = \emptyset$. Therefore $A(R) = A(R_1) + A(R_2)$. This shows that we will need two integrals.

$$\begin{aligned}A(R) = A(R_1) + A(R_2) &= \int_{-3}^{-2}\left(x^2-4-(-x^2-2x)\right)\mathrm{d}x + \int_{-2}^{1}\left(-x^2-2x-(x^2-4)\right)\mathrm{d}x \\ &= \int_{-3}^{-2}\left(2x^2+2x-4\right)\mathrm{d}x + \int_{-2}^{1}\left(-2x^2-2x+4\right)\mathrm{d}x \\ &= \left[\frac{2}{3}x^3+x^2-4x\right]_{-3}^{-2} + \left[-\frac{2}{3}x^3-x^2+4x\right]_{-2}^{1} \\ &= \left(-\frac{16}{3}+4+8\right) - (-18+9+12) + \left(-\frac{2}{3}-1+4\right) - \left(\frac{16}{3}-4-8\right) \\ &= -\frac{34}{3}+24 = \boxed{\frac{38}{3}}\end{aligned}$$

p.298, pr.36



5. (a) [10 Points] Figure shows two right circular cones, one upside down inside the other. The two bases are parallel, and the vertex of the smaller cone lies at the center of the larger cone's base. What values of r and h will give the smaller cone the largest possible volume?



p.258, pr.56

Solution: Let h be the height and r be the base radius for the inner cone. From the figure we get

$$\frac{r}{12-h} = \frac{6}{12} \Rightarrow 2r+h = 12.$$

Now we need to find for what values of h and r, the inner cone will have largest volume i.e., we need to maximize $V = \frac{1}{3}\pi r^2 h$.

We need to find the absolute maximum value of

$$V(r) = \frac{1}{3}\pi r^2(12-2r) = \frac{2\pi}{3} r^2(6-r) \textbf{ over } [0,6].$$

Take the derivative and set equal to zero to find the critical point(s).

$$V'(r) = \frac{2\pi}{3}\left(2r(6-r)+r^2(-1)\right) = 0\frac{2\pi}{3}\left(12r-2r^2-r^2\right) \quad = 0$$

$$\frac{2\pi}{3}\left(12r-3r^2\right) = 0$$

$$r(12-3r) = 0$$

$$r = 0, r = 4 \textbf{ critical points}$$

We have:

$V(0) = 0$ **ABS. MIN.**

$V(4) = \frac{2\pi}{3}(4^2)(6-4) = \frac{64\pi}{3}$ **ABS. MAX.**

$V(6) = 0$ **ABS. MIN.**

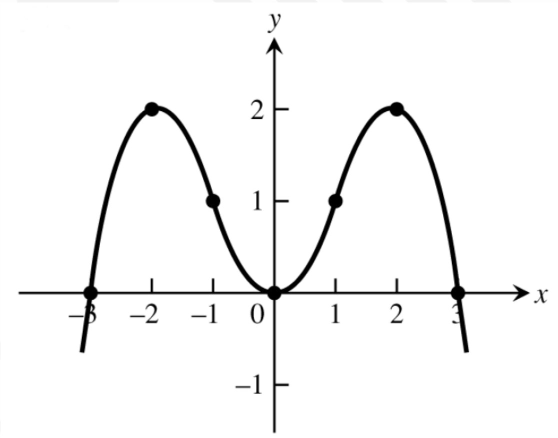
p.583, pr.32

(b) 10 Points Using the following properties of a twice-differentiable function $y = f(x)$, sketch its graph on the grid by indicating all significant points. Write the intervals of increase/decrease and concave up/down.

x	y	**Derivatives / Türevler**	
$x=-3$	0	$y'>0,$	$y''<0$
$x<-2$		$y'>0,$	$y''<0$
-2	2	$y'=0,$	$y''<0$
$-2<x<-1$		$y'<0,$	$y''>0$
-1	1	$y'<0,$	$y''=0$
$-1<x<0$		$y'<0,$	$y''>0$
0	0	$y'=0,$	$y''>0$
$0<x<1$		$y'>0,$	$y''>0$
1	1	$y'>0,$	$y''=0$
$1<x<2$		$y'>0,$	$y''<0$
2	2	$y'=0,$	$y''<0$
$x>2$		$y'<0,$	$y''<0$
$x=3$	0	$y'<0,$	$y''<0$

Solution: The given table indicates that the graph

- is increasing and concave down on $(-\infty,-2)$
- has a local maximum at $(-2,2)$
- decreasing and concave down on $(-2,-1)$
- has a point of inflection at $(-1,1)$
- decreasing and concave up on $(-1,0)$
- has a local minimum at $(0,0)$
- is increasing and concave up on $(0,1)$
- has a point of inflection at $(1,1)$
- is increasing and concave down on $(1,2)$
- has a local maximum at $(2,2)$
- decreasing and concave down on $(2,+\infty)$



p.452, pr.24