Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- This exam has 3 pages plus this cover sheet and 6 problems. Please make sure that your exam is complete.
- Time limit is 90 min.

Problem	Points	Score
1	20	
2	15	
3	15	
4	15	
5	20	
6	15	
Total:	100	

Do not write in the table to the right.

1. Suppose $f(x) = x^4 - 2x^2$ p.211, pr.17

(a) $\boxed{\text{2 Points}}$ Identify the domain of f and any symmetries the curve may have.

Solution: The domain of f is $(-\infty, \infty)$ and the graph is symmetric with respect to the y-axis since $f(-x) = (-x)^4 - 2(-x)^2 = x^4 - 2x^2 = f(x)$.

(b) $\boxed{\text{3 Points}}$ Find f' and f''.

Solution: Differentiation gives $f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1)$ and $f''(x) = 12x^2 - 4 = 4(3x^2-1) = 12(x - \frac{1}{\sqrt{3}})(x + \frac{1}{\sqrt{3}})$.

(c) $\boxed{\text{3 Points}}$ Find the critical points of f, if any, and identify the function's behavior at each one.

Solution: Solving $f'(x) = 0$ gives us the critical numbers 0, -1, and 1. By the Second Derivative Test $f''(0) = -4 < 0$ implies that the graph has a Local Maximum at $x = 0$ and $f''(\pm 1) = 8 > 0$ implies that the graph has a Local Minimum at $x = \pm 1$. Hence the points $(\pm 1, -1)$ are points of Local Minima whereas $(0,0)$ is the only point of Local Maximum.

(d) $\boxed{\text{4 Points}}$ Find where the curve is increasing and where it is decreasing.

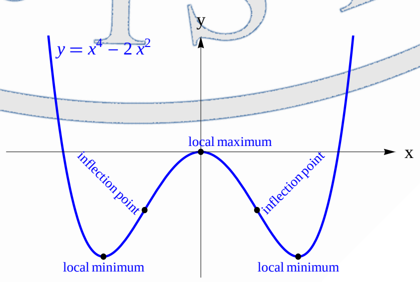
Solution: The critical points split the real line into 4 open subintervals, namely, $(-\infty,-1)$, $(-1,0)$, $(0,1)$, and $(1,\infty)$. By considering test values on each of these intervals, we see that f is increasing on $(-1,0) \cup (1,\infty)$ and decreasing on $(-\infty,-1) \cup (0,1)$.

(e) $\boxed{\text{4 Points}}$ Find the points of inflection, if any occur, and determine the concavity of the curve.

Solution: Since $f''(x) = 0$ implies $x = \pm\frac{1}{\sqrt{3}}$, there are two points of inflection for the graph of f. By looking at the sign for f'', we see that the graph is concave up on $(-\infty, -\frac{1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, +\infty)$ and concave down on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. Hence $(\pm\frac{1}{\sqrt{3}}, -\frac{5}{9})$ are the two inflection points.

(f) $\boxed{\text{4 Points}}$ Sketch the graph of f using your results in (a), (b), (c), (d) and (e).

Solution:



2. $\boxed{\text{15 Points}}$ Use only optimization to find the point on the line $\frac{x}{a}+\frac{y}{b}=1$ that is closest to the origin. p.222, pr.28

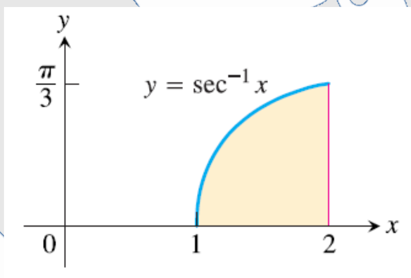
Solution: Suppose $P(x,y)$ is the point on this line that is closest to the origin. Let $d=\sqrt{(x-0)^2+(y-0)^2}=\sqrt{x^2+y^2}$ and $\frac{x}{a}+\frac{y}{b}=1 \Rightarrow y=-\frac{b}{a}x+b$.

We can minimize d by minimizing $D=\left(\sqrt{x^2+y^2}\right)^2=x^2+\left(-\frac{b}{a}x+b\right)^2 \Rightarrow D'=2x+2\left(-\frac{b}{a}x+b\right)\left(-\frac{b}{a}\right)=2x+\frac{2b^2}{a^2}x-\frac{2b^2}{a}$. Hence $D'=0 \Rightarrow 2\left(x+\frac{b^2}{a^2}x-\frac{b^2}{a}\right)=0 \Rightarrow x=\frac{ab^2}{a^2+b^2}$ is the critical point $\Rightarrow y=-\frac{b}{a}\left(\frac{ab^2}{a^2+b^2}\right)+b=\frac{a^2b}{a^2+b^2}$. Thus $D''=2+\frac{2b^2}{a^2} \Rightarrow D''\left(\frac{ab^2}{a^2+b^2}\right)=2+\frac{2b^2}{a^2}>0 \Rightarrow$ the critical point is local minimum by the Second Derivative Test, $\Rightarrow \left(\frac{ab^2}{a^2+b^2},\frac{a^2b}{a^2+b^2}\right)$ is the point on the line $\frac{x}{a}+\frac{y}{b}=1$ that s closest to the origin.

3. $\boxed{\text{15 Points}}$ The region between the curve $y=\sec^{-1}x$ and the x-axis from $x=1$ to $x=2$ (shown here) is revolved about y-axis to generate a solid. Find the volume of the solid. p.414, pr.108

Solution:

$$\begin{aligned} V &= \pi\int_0^{\pi/3}\left[2^2-(\sec y)^2\right]dx \\ &= \pi\left[4y-\tan y\right]_0^{\pi/3} \\ &= \pi\left[\frac{4\pi}{3}-\sqrt{3}\right] \end{aligned}$$



4. $\boxed{\text{15 Points}}$ Find the area of the surface generated by revolving the curve about the y-axis.
$x=\sqrt{2y-1}$, $5/8\le y\le 1$ p.336 pr.20

Solution:

$$x = \sqrt{2y-1} \Rightarrow \frac{dx}{dy}=\frac{1}{\sqrt{2y-1}} \Rightarrow \left(\frac{dx}{dy}\right)^2=\frac{1}{2y-1} \Rightarrow S=\int_{5/8}^{1}2\pi\sqrt{2y-1}\sqrt{1+\frac{1}{2y-1}}\,dy$$

$$S = 2\pi\int_{5/8}^{1}\sqrt{(2y-1)+1}\,dy=2\pi\sqrt{2}\int_{5/8}^{1}y^{1/2}\,dy=2\pi\sqrt{2}\left[\frac{1}{3}y^{3/2}\right]_{5/8}^{1}$$

$$\longrightarrow \quad S=\frac{4\pi\sqrt{2}}{3}\left[1^{3/2}-\left(\frac{5}{8}\right)^{3/2}\right]=\frac{\pi}{12}\left(16\sqrt{2}-5\sqrt{5}\right)$$

5. (a) $\boxed{\text{10 Points}}$ $\int 2^{\tan x} \sec^2 x \, dx = ?$ p.430, pr.50

Solution: Let $u = \tan x$. Then $du = \sec^2 x \, dx$. Hence we have

$$\int 2^{\tan x} \sec^2 x \, dx = \int 2^u \, du = \frac{2^u}{\ln 2} + C = \frac{2^{\tan x}}{\ln 2} + C$$

(b) $\boxed{\text{10 Points}}$ $\lim_{x \to 0} \frac{\sin 3x - 3x + x^2}{\sin x \sin 2x} = ?$ p.402, pr.50

Solution: This has the indeterminate form $\frac{0}{0}$. Hence the L'Hôpital's Rule applies.

$$\lim_{x \to 0} \frac{\sin 3x - 3x + x^2}{\sin x \sin 2x} \stackrel{\text{L'H}}{=} \lim_{x \to 0} \frac{3\cos 3x - 3 + 2x}{2\sin x \cos 2x + \cos x \sin 2x} = \lim_{x \to 0} \frac{3\cos 3x - 3 + 2x}{\sin x \cos 2x + \sin 3x}$$

$$\stackrel{\text{L'H}}{=} \lim_{x \to 0} \frac{-9\sin 3x + 2}{-2\sin x \sin 2x + \cos x \cos 2x + 3\cos 3x} = \frac{2}{4} = \frac{1}{2}$$

6. $\boxed{\text{15 Points}}$ $\int_{-2}^{2} \frac{3\,dt}{4 + 3t^2} = ?$ p.431, pr.67

Solution: First notice that

$$\int_{-2}^{2} \frac{3\,dt}{4 + 3t^2} = \frac{3}{4} \int_{-2}^{2} \frac{dt}{1 + \frac{3t^2}{4}} = \frac{3}{4} \int_{-2}^{2} \frac{dt}{1 + (\frac{\sqrt{3}t}{2})^2}.$$

We use the substitution $u = \frac{\sqrt{3}}{2} t$ and so $du = \frac{\sqrt{3}}{2} dt$. When $t = -2$, we have $u = -\sqrt{3}$ and when $t = 2$, we have $u = \sqrt{3}$.

$$\begin{aligned}
\int_{-2}^{2} \frac{3\,dt}{4 + 3t^2} &= \frac{3}{4} \int_{-2}^{2} \frac{dt}{1 + \frac{3t^2}{4}} \\
&= \frac{3}{4} \int_{-2}^{2} \frac{dt}{1 + (\frac{\sqrt{3}}{2} t)^2} \\
&= \frac{3}{4} \frac{2}{\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} \frac{du}{1 + u^2} \\
&= \left[\frac{\sqrt{3}}{2} \tan^{-1} u \right]_{-\sqrt{3}}^{\sqrt{3}} = \frac{\sqrt{3}}{2} \left[\tan^{-1}(\sqrt{3}) - \tan^{-1}(-\sqrt{3}) \right] \\
&= \frac{\sqrt{3}}{2} \left[\frac{\pi}{3} - \left(-\frac{\pi}{3} \right) \right] = \frac{\pi}{\sqrt{3}}.
\end{aligned}$$