Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives.**
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 100 min.

Do not write in the table to the right.

Problem	Points	Score
1	14	
2	24	
3	14	
4	20	
5	15	
6	13	
Total:	100	

1. (a) 7 Points $\lim_{x\to 3} \frac{x^2-9}{\sqrt{x^2+7}-4} = ?$ **(Do not use L'Hôpital's Rule/L'Hôpital Kuralı kabul edilmeyecektir.)**

Solution:

$$\lim_{x\to 3} \frac{x^2-9}{\sqrt{x^2+7}-4} = \lim_{x\to 3} \frac{x^2-9}{\sqrt{x^2+7}-4}\,\frac{\sqrt{x^2+7}+4}{\sqrt{x^2+7}+4} = \lim_{x\to 3} \frac{(x^2-9)(\sqrt{x^2+7}+4)}{x^2+7-16}$$

$$= \lim_{x\to 3} \frac{\cancel{(x^2-9)}(\sqrt{x^2+7}+4)}{\cancel{(x^2-9)}}$$

$$= \lim_{x\to 3}(\sqrt{x^2+7}+4) = \sqrt{3^2+7}+4 = \boxed{8}$$

p.57, pr.86

(b) 7 Points $\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\,dx = ?$

Solution: Let $u = 1+\sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\,dx$, so $2du = \frac{dx}{\sqrt{x}}$. Thus

$$\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\,dx = \int 2u^{1/3}\,du = 2\left[\frac{u^{1/3+1}}{1/3+1}\right] + C = \frac{3}{2}u^{4/3} + C = \boxed{\frac{3}{2}(1+\sqrt{x})^{4/3} + C}$$

p.290, pr.6

2. Given the curve $y = \dfrac{4x}{x^2+4}$ and derivatives $y' = \dfrac{-4x^2+16}{(x^2+4)^2}$ and $y'' = \dfrac{8x^3-96x}{(x^2+4)^3}$

(a) [3 Points] Identify the *domain* of f and any *symmetries* the curve may have.

Solution: Domain is $(-\infty,+\infty)$. Since $y(-x) = \dfrac{4(-x)}{(-x)^2+4} = -\dfrac{4x}{x^2+4} = -y(x)$ for each $x \in (-\infty,+\infty)$, the function is odd and so graph is symmetric with respect to the origin.

p.212, pr.92

(b) [5 Points] Find the intervals where the graph is increasing and decreasing. Find the local maximum and minimum values.

Solution:
$y' = \dfrac{-4x^2+16}{(x^2+4)^2} > 0$ if and only if $-4x^2+16 > 0$, that is iff $x^2 < 4$, i.e., graph is increasing on $(-2,+2)$ $-2 < x < 2$ and decreasing on $(-\infty,-2)\cup(2,+\infty)$.. By the First Derivative Test, graph has a local minimum at $x=-2$ and a local maximum at $x=2$. The local minimum value is $f(-2) = -1$ which is the absolute mi. value and local maximum value is $f(2) = 1$ which is the absolute max. value. p.241, pr.45

(c) [5 Points] Determine where the graph is concave up and concave down, and find any inflection points.

Solution:
Notice that $y'' = \dfrac{8x^3-96x}{(x^2+4)^3} > 0$ if and only if $8x^3-96x > 0$, that is iff $x(x^2-12) > 0$. solving the last inequality, we see that graph is concave up on $(-2\sqrt{3},0)\cup(2\sqrt{3},+\infty)$ and concave down on $(-\infty,-2\sqrt{3})\cup(0,2\sqrt{3})$. Now due to the sign changes in y'', there are three points of inflection namely, $(-2\sqrt{3},-\sqrt{3}/2)$, $(0,0)$, and $(2\sqrt{3},\sqrt{3}/2))$.

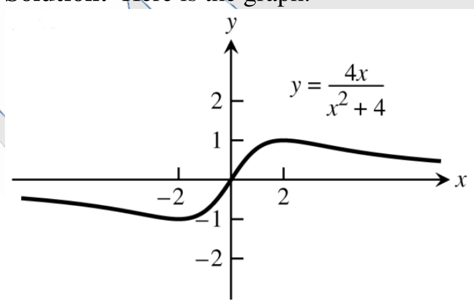
p.241, pr.45

(d) [5 Points] Find the asymptotes.

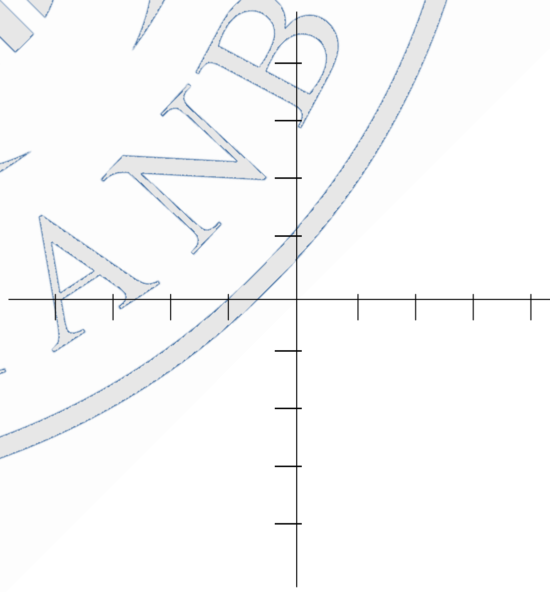
Solution:
Since $x^2+4 \neq 0$, for each $x \in (-\infty,\infty)$, graph can not have a vertical asymptote. Since $\displaystyle\lim_{x\to\pm\infty} \frac{4x}{x^2+4} = 0$, we see that $y=0$ is a horizontal asymptote. p.241, pr.45

(e) [6 Points] Draw the graph of f showing all significant features.

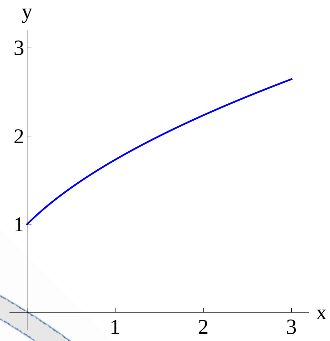
Solution: Here is the graph.



p.241, pr.45



3. [14 Points] Find the area of the surface generated by revolving the curve about the x-axis. $y=\sqrt{2x+1}$, $0\le x\le 3$.



Solution: $\frac{dy}{dx} = \frac{d}{dx}(\sqrt{2x+1}) = \frac{1}{\sqrt{2x+1}} \Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{1}{\sqrt{2x+1}}\right)^2} = \sqrt{\frac{2x+2}{2x+1}}$.

Now $y\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{2x+1}\sqrt{\frac{2x+2}{2x+1}} = \sqrt{2x+2}$. Hence the area of the surface of revolution is

$$S = 2\pi\int_0^3 y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\sqrt{2}\pi\int_0^3\sqrt{x+1}\,dx$$
$$= 2\sqrt{2}\pi\left[\frac{2}{3}(x+1)^{3/2}\right]_0^3 = 2\sqrt{2}\pi\frac{2}{3}(8-1) = \boxed{\frac{28\pi\sqrt{2}}{3}}$$

p.241, pr.45

4. (a) [12 Points] Find the length of the curve $y=\int_0^x\sqrt{\cos 2t}\,dt$ from $x=0$ to $x=\pi/4$.

Solution: By Fundamental Theorem of Calculus Part I, we have $\frac{dy}{dx} = \frac{d}{dx}\left(\int_0^x\sqrt{\cos 2t}\,dt\right) = \sqrt{\cos 2x}$. Thus for $x\in[0,\pi/4]$, we have

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\sqrt{\cos 2x}\right)^2} = \sqrt{1+\cos 2x} = \sqrt{2\cos^2 x} = \sqrt{2}|\cos x| = \sqrt{2}\cos x.$$

Hence the length we want is then

$$L_0^{\pi/4} = \int_0^{\pi/4}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{\pi/4}\sqrt{2}\cos x\,dx = \sqrt{2}[\sin x]_0^{\pi/4} = \sqrt{2}(\sin(\pi/4)-\sin(0)) = \sqrt{2}\frac{1}{\sqrt{2}} = \boxed{1}$$

p.191, pr.54

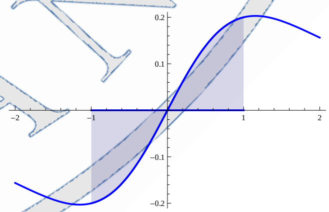
(b) [8 Points] $\int_{-1}^{1}\frac{5r}{(4+r^2)^2}\,dr = ?$

Solution:

Let $u=4+r^2$. Then $du=2r\,dr$. Hence $\frac{1}{2}du = r\,dr$. When $r=-1$, we have $u=5$ and when $r=1$, we have $u=5$. Thus, we have

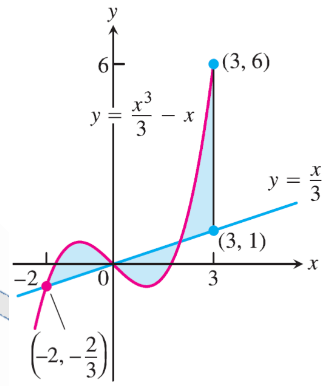
$$\int_{-1}^{1}\frac{5r}{(4+r^2)^2}\,dr = \int_5^5\frac{1}{2}u^{-2}\,du = \boxed{0}$$

See the figure on the right. The graph is symmetric about the origin and so the equal signed areas cancel each other giving that the integral equal to zero.



p.191, pr.54

5. 15 Points Find the total area of the shaded region.



Solution: AREA $= A_1 + A_2 + A_3$. Let $f(x) := \frac{x^3}{3} - x$ and let $g(x) := \frac{x}{3}$. Then we have

A_1: For the sketch given, $f(x) \geq g(x)$, $a = -2$, and $b = 0$ and: $f(x) - g(x) = \left(\frac{x^3}{3} - x\right) - \frac{x}{3} = \frac{x^3}{3} - \frac{4}{3}x = \frac{1}{3}(x^3 - 4x)$ and so

$$\begin{aligned} A_1 &= \frac{1}{3}\int_{-2}^{0}(x^3 - 4x)\,dx = \frac{1}{3}\left[\frac{1}{4}x^4 - 2x^2\right]_{-2}^{0} \\ &= 0 - \frac{1}{3}(4 - 8) = \boxed{\frac{4}{3}} \end{aligned}$$

A_2: For the sketch given, $a = 0$ and we find b by solving the equations $y = \frac{x^3}{3} - x$ and $y = \frac{x}{3}$ simultaneously for x:

$$\frac{x^3}{3} - x = \frac{x}{3} \Rightarrow \frac{x^3}{3} - \frac{4}{3}x = 0 \Rightarrow \frac{x}{3}(x-2)(x+2) = 0 \Rightarrow x = \pm 2, x = 0.$$

Thus $b = 2 : g(x) - f(x) = -\frac{1}{3}(x^3 - 4x)$ and so

$$\begin{aligned} A_2 &= -\frac{1}{3}\int_{0}^{2}(x^3 - 4x)\,dx = \frac{1}{3}\int_{0}^{2}(4x - x^3)\,dx = \frac{1}{3}\left[2x^2 - \frac{1}{4}x^4\right]_{0}^{2} \\ &= \frac{1}{3}(8 - 4) = \frac{4}{3}; \end{aligned}$$

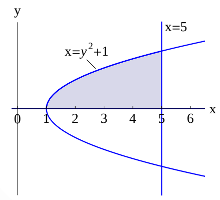
A_3: For the sketch given, $a = 2$, and $b = 3$: $f(x) - g(x) = \left(\frac{x^3}{3} - x\right) - \frac{x}{3} = \frac{x^3}{3} - \frac{4}{3}x = \frac{1}{3}(x^3 - 4x)$ and so

$$\begin{aligned} A_3 &= \frac{1}{3}\int_{2}^{3}(x^3 - 4x)\,dx = \frac{1}{3}\left[\frac{1}{4}x^4 - 2x^2\right]_{2}^{3} \\ &= \frac{1}{3}\left(\frac{81}{4} - (2)(9) - \left(\frac{16}{4} - 8\right)\right) = \frac{1}{3}\left(\frac{81}{4} - 14\right) = \frac{25}{12} \end{aligned}$$

Therefore, AREA $= A_1 + A_2 + A_3 = \frac{4}{3} + \frac{4}{3} + \frac{25}{12} = \frac{32 + 25}{12} = \boxed{\frac{19}{4}}$

p.298, pr.40

6. 13 Points Find the volume of the solid generated by revolving the region about the x-axis bounded by
$x = y^2 + 1$, $x = 5$, $y = 0$, and $y \geq 0$.



Solution: If $0 \leq y \leq 2$, then a horizontal strip of the given region "at" y has length $5-(y^2+1)$ and moves around a circle of radius y, so the volume generated rotation of that region around x-axis is

$$\begin{aligned} V &= \int_c^d 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dy = \int_0^2 2\pi y(5-y^2-1)\,dy \\ &= 2\pi\int_0^2 (4y-y^3)\,dy \\ &= 2\pi\left[2y^2-\frac{1}{4}y^4\right]_0^2 = 2\pi(8-4) = 8\pi. \end{aligned}$$

Alternatively, when the method of disks is used, one gets

$$\begin{aligned} V &= \int_1^5 \pi[R(x)]^2\,dx = \pi\int_1^5\left[\sqrt{x-1}\right]^2 dx \\ &= \pi\int_1^5 (x-1)\,dx \\ &= \pi\left[\frac{1}{2}x^2-x\right]_1^5 \\ &= \pi\left[\left(\frac{25}{2}-5\right)-\left(\frac{1}{2}-1\right)\right] \\ &= \boxed{8\pi} \end{aligned}$$

p.236, pr.27