Math 113 Summer 2016 **Final Exam** August 11, 2016

Your Name / Ad - Soyad (**75 min.**) Signature / İmza

Student ID # / Öğrenci No (**mavi tükenmez!**)

Problem	1	2	3	4	Total
Points:	32	20	25	23	100
Score:					

You have **75 minutes**. **(Cell phones off and away!)**. No books, notes or calculators are permitted. **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

---

1. (a) (8 Points) Evaluate the $\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta$.

Solution: First let $u = \sin\sqrt{\theta}$. Then $du = \frac{1}{2\sqrt{\theta}}\cos\sqrt{\theta}\,d\theta$. Therefore

$$\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta = 2\int \underbrace{\frac{1}{\sin^2\sqrt{\theta}}}_{1/u^2}\underbrace{\frac{1}{2\sqrt{\theta}}\cos\sqrt{\theta}\,d\theta}_{du}$$
$$= 2\int \frac{1}{u^2}\,du$$
$$= 2\frac{-1}{u} + C = \boxed{-\frac{2}{\sin\sqrt{\theta}} + C}$$

p.290, pr.36

(b) (7 Points) Find the derivative $\frac{dy}{dx}$ if $y = \int_{\tan x}^{0} \frac{dt}{1+t^2}$.

Solution: First let $u = \tan x$ and so $du = \sec^2 x\, dx$. Now we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\int_{\tan x}^{0}\frac{dt}{1+t^2}\right)$$
$$= \frac{d}{dx}\left(-\int_{0}^{\tan x}\frac{dt}{1+t^2}\right)$$
$$= -\frac{d}{dx}\left(\int_{0}^{u}\frac{dt}{1+t^2}\right)$$
$$= -\frac{d}{du}\left(\int_{0}^{u}\frac{dt}{1+t^2}\right)\frac{du}{dx}$$
$$= -\frac{1}{1+u^2}\sec^2 x$$
$$= -\frac{1}{1+\tan^2 x}\sec^2 x = -\frac{1}{\cancel{\sec^2 x}}\cancel{\sec^2 x} = \boxed{-1}$$

p.282 pr.40

(c) (7 Points) Find the limit $\lim_{x\to 0}\frac{\frac{1}{x-1}+\frac{1}{x+1}}{x}$.

You are not allowed to use L'Hôpital's Rule.

Solution: We first clear the fractions and then compute the limit.

$$\lim_{x\to 0}\frac{\frac{1}{x-1}+\frac{1}{x+1}}{x} = \lim_{x\to 0}\left(\frac{x+1+(x-1)}{(x-1)(x+1)}\right)\left(\frac{1}{x}\right)$$
$$= \lim_{x\to 0}\frac{2\cancel{x}}{(x^2-1)\cancel{x}}$$
$$= \lim_{x\to 0}\frac{2}{(x^2-1)} = \frac{2}{(0^2-1)} = \boxed{-2}$$

p.55 pr.32

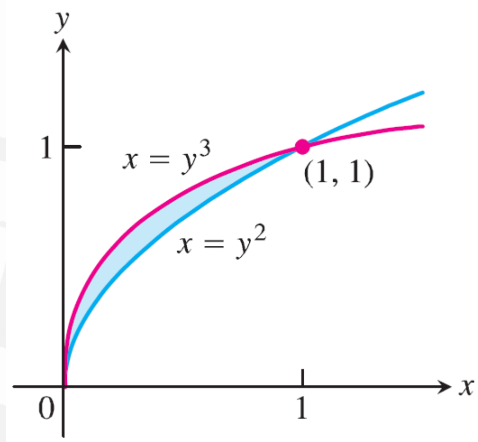
(d) (10 Points) Find $\frac{dy}{dt}$ if $y = 3t\left(2t^2-5\right)^4$.

Solution: By the product rule for derivatives, we have

$$\frac{dy}{dt} = (3t)\frac{d}{dt}\left[\left(2t^2-5\right)^4\right] + \left(2t^2-5\right)^4\frac{d}{dt}(3t)$$
$$= (3t)(4)\left(2t^2-5\right)^3\frac{d}{dt}(2t^2-5) + \left(2t^2-5\right)^4(3)$$
$$= 3\left(2t^2-5\right)^3\left((4t)(4t)+3(2t^2-5)\right)$$
$$= \boxed{3\left(2t^2-5\right)^3\left(22t^2-15\right)}$$

p.147, pr.57

2. (a) (10 Points) Find the total area of the shaded region.



Solution: One can find this area by integrating with respect to either x or y. First integrating with respect to x gives

$$\begin{aligned} A &= \int_0^1 \left(x^{1/3} - x^{1/2}\right)\, dx \\ &= \left[\frac{x^{4/3}}{4/3} - \frac{x^{3/2}}{3/2}\right]_0^1 \\ &= \frac{3}{4} - \frac{2}{3} = \boxed{\frac{1}{12}} \end{aligned}$$

Much easier is to integrate with respect to y.

$$\begin{aligned} A &= \int_0^1 \left(y^2 - y^3\right)\, dy \\ &= \left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 \\ &= \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}} \end{aligned}$$

p.298, pr.32

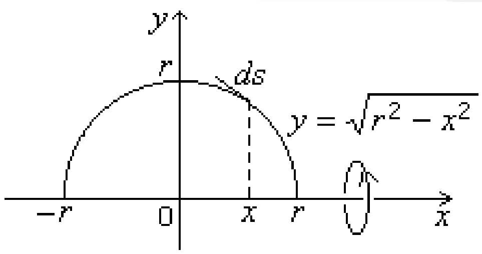
(b) (10 Points) Show that the surface area of a sphere of r radius is $4\pi r^2$ by finding the area of the surface generated by revolving the curve $y = \sqrt{r^2 - x^2}$, $-r \le x \le r$ about the x-axis.

Solution:
For simplicity, compute the derivative and its square.

$$\begin{aligned} y = \sqrt{r^2 - x^2} \Rightarrow \frac{dy}{dx} &= \frac{-2x}{2\sqrt{r^2 - x^2}} \\ &= \frac{-x}{\sqrt{r^2 - x^2}} \\ \Rightarrow \left(\frac{dy}{dx}\right)^2 &= \frac{x^2}{r^2 - x^2} \\ \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2} \\ \Rightarrow y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{r^2 - x^2}\sqrt{\frac{r^2}{r^2 - x^2}} = \sqrt{r^2} = |r| = r \end{aligned}$$

$$S = 2\pi \int_{-r}^{r} y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_{-r}^{r} r\, dx = 2\pi [rx]_{-r}^{r} = 2\pi r(r - (-r)) = \boxed{4\pi r^2}$$

p.336, pr.25



3. (a) (12 Points) Find the volume of the solid generated by revolving the shaded region about the x-axis.

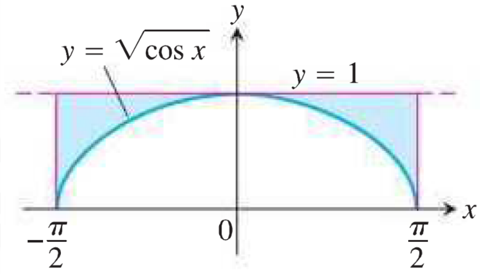
Solution: We wish to use the method of washers. For that purpose, we *slice vertically*. Hence here outer radius is $R(x) = 1$ and inner radius is $r(x) = \sqrt{\cos x}$. Hence the required volume is

$$V = \pi \int_{-\pi/2}^{\pi/2} \left([R(x)]^2 - [r(x)]^2\right) \, dx = \pi \int_{-\pi/2}^{\pi/2} \left(1^2 - (\sqrt{\cos x})^2\right) \, dx$$

$$= \pi \int_{-\pi/2}^{\pi/2} \underbrace{(1 - \cos x)}_{\text{even function}} \, dx$$

$$= 2\pi \int (1 - \cos x) \, dx = 2\pi[x - \sin x]_0^{\pi/2} = 2\pi \left(\frac{\pi}{2} - 1\right) = \boxed{\pi^2 - 2\pi}$$

p.317, pr.33



(b) (13 Points) Use the *shell method* to find the volume of the solid generated by revolving the shaded region about the y-axis.

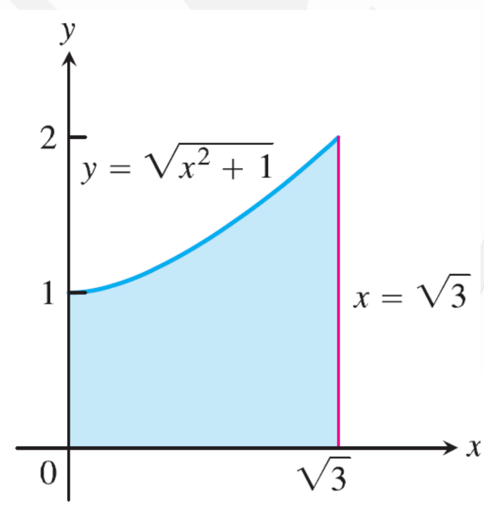
Solution: As asked in the question, we have to employ the method of shells. For this, we must slice vertically. If $0 \leq x \leq \sqrt{3}$, then a vertical strip of the given region "at" x has length $\sqrt{x^2+1}$ and moves around a circle of radius x, so the volume generated by rotation of that region around y-axis is

$$V = \int_c^d 2\pi \begin{pmatrix}\text{shell} \\ \text{radius}\end{pmatrix} \begin{pmatrix}\text{shell} \\ \text{height}\end{pmatrix} \, dx = \int_{x=0}^{x=\sqrt{3}} \pi \underbrace{(\sqrt{x^2+1})}_{\sqrt{u}} \underbrace{2x \, dx}_{du}$$

$$= \pi \int_{u=1}^{u=4} u^{1/2} \, du$$

$$= \pi \left[\frac{2}{3} u^{3/2}\right]_1^4 = \frac{2\pi}{3}(4^{3/2} - 1) = \left(\frac{2\pi}{3}\right)(8 - 1) = \boxed{\frac{14\pi}{3}}.$$

p.324, pr.5



4. Consider the function $y = -2x^3 + 6x^2 - 3$.

(a) (5 Points) Identify the *domain* of f and any *symmetries* the curve may have.

Solution: This is a polynomial function so the domain is $(-\infty, \infty)$. Next notice that $y(-x) = -2(-x)^3 + 6(-x)^2 - 3 = 2x^3 + 6x^2 -$. This quantity clearly can neither equal $y(x)$ nor $-y(x)$ *for all* x. This shows that the graph is not symmetric with respect to y-axis and not symmetric with respect to the origin. We conclude that there is no possible symmetry for the graph.

p.211, pr.33

(b) (5 Points) Find the *intervals* where the graph is *increasing* and *decreasing*. Find the *local maximum* and *minimum* values.

Solution: The first derivative is $y' = -6x^2 + 12x = 6x(-x+2)$. The only critical numbers are $x = 0$ and $x = 2$. This brings about three intervals, namely $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$. On the intervals $(-\infty, 0)$ and $(2, \infty)$ graph is decreasing and is increasing on $(0, 2)$. Furthermore, the point $(0, -3)$ is a local minimum and at the point $(2, 5)$ graph has a local maximum.

$$\begin{array}{ccccc} - & & + & & - \\ \hline & 0 & & 2 & \end{array} \quad y'$$

p.211, pr.33

(c) (5 Points) Determine where the graph is *concave up* and *concave down*, and find any *inflection points*.

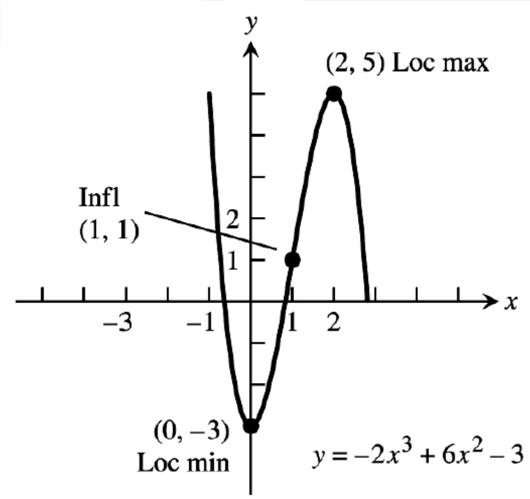
Solution: Now the second derivative is $y'' = -12x + 12$ which is zero only when $x = 1$. So this splits the real line into two subintervals: $(-\infty, 1)$ and $(1, \infty)$. Since over $(-\infty, 1)$, positive and on the second interval $(1, \infty)$, y'' is negative, we conclude that the graph is concave up on $(-\infty, 1)$ and concave down on $(1, \infty)$. Moreover the point $(1, 1)$ is a point of inflection.

$$\begin{array}{ccccc} + & + & & - & - \\ \hline & & 1 & & \end{array} \quad y''$$

p.211, pr.33

(d) (8 Points) *Sketch the graph* of the function. Label the extreme points and the inflection points.

Solution:



p.211, pr.33