Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

Do not write in the table to the right.

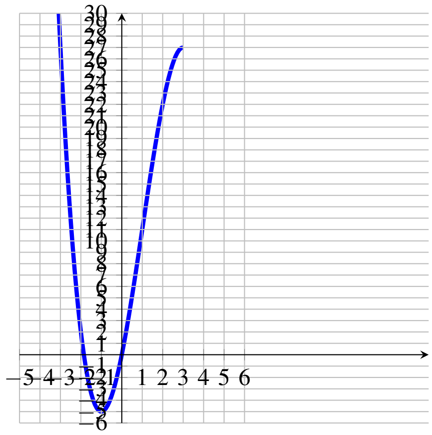
Problem	Points	Score
1	14	
2	14	
3	14	
4	17	
5	16	
6	11	
7	14	
Total:	100	

1. 14 Points Determine the values of constants a, b, c, and d so that $f(x) = ax^3 + bx^2 + cx + d$ has a local maximum at $(0,0)$ and a local minimum at the point $(1,-1)$.

Solution: $f(x) = ax^3 + bx^2 + cx + d \Rightarrow f'(x) = 3ax^2 + 2bx + c$ and $f''(x) = 6ax + 2b$. Since there is a local minimum at $x = 1$ we have $f'(1) = 0 \Rightarrow 3a + 2b + c = 0$. Similarly, local maximum at $x = 0$ implies $f'(0) = 0 \Rightarrow 3a(0)^2 + 2b(0) + c = 0$ and so $c = 0$. Furthermore, the graph passes through $(1,-1)$ implies $f(1) = -1 \Rightarrow a + b + c + d = -1$ and passes through $(0,0)$ implies $f(0) = 0 \Rightarrow d = 0$. Now we have the system

$$\begin{aligned} a + b &= -1 & (1) \\ 3a + 2b &= 0. & (2) \end{aligned}$$

Solving the system, we have $\boxed{a = 2, b = -3, c = 0, d = 0}$ so there is only one curve satisfying the requirements which is $f(x) = 2x^3 - 3x^2$. Just to check that this is the correct curve we need, we employ the Second Derivative Test here (indeed, this is necessary here) $f''(0) = 12(0) - 6 = -6 < 0$ so local max. at $x = 0$ and $f''(1) = 12(1) - 6 = 6 > 0$ so local min. at $x = 1$.



Therefore $\boxed{f(x) = 2x^3 - 3x^2}$.

p.203, pr.70

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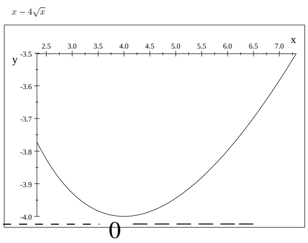
2. [14 Points] Find the extreme values (absolute and local) of $f(x) = x - 4\sqrt{x}$ and where they occur.

Solution: First the domain of f is $[0, +\infty)$. Note that

$$y' = 1 - \frac{4}{2\sqrt{x}} = \frac{\sqrt{x} - 2}{\sqrt{x}},$$

which is zero at $x = 4$ and is undefined when $x = 0$. If $x \in (0, 4)$, we have $y' < 0$ and if $x \in (4, +\infty)$, we have $y' > 0$. Hence there is a *local minimum* at $x = 4$. Now for the absolute extrema, we compare the values of f at 0 and 4 respectively. Hence $f(0) = 0 - 4\sqrt{0} = 0$ is the *local maximum* and $f(4) = 4 - 4\sqrt{4} = \boxed{-4}$ is the *absolute minimum*.



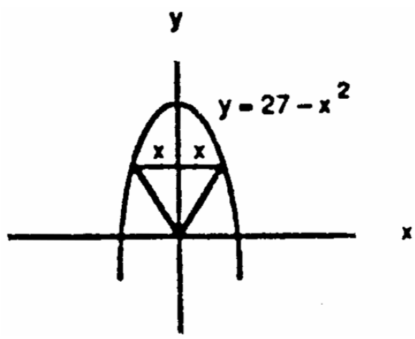
sign of y': $- - - - - - -$ 0 $+ + + + + + + + + + + + + +$ $- - - - - - - - -$ 0 $- - - - - -$

x: 0, 0, 4, $+\infty$

graph of y: concave down | concave up | concave down

p.191, pr.54

3. 14 Points An isosceles triangle has its vertex at the origin and its base parallel to the x-axis with the vertices above the axis on the curve $y = 27 - x^2$. Find the largest area the triangle can have.



Solution: The area of the largest isosceles triangle that can be drawn with one vertex at the origin and with the others on a line parallel to and above the x-axis and on the curve $y = 27 - x^2$ is: If we let x be the distance x-distance between the vertex and the other vertex to the right (x would be half its base, and it doesn't matter which vertex we choose since $y = 27 - x^2$ is symmetric about the y-axis), then the height will be the x-value at that point; $27 - x^2$. So the total area would be: $\frac{1}{2}A = (\text{base})(\text{height}) = (2x)(27 - x^2)/2 = 27x - x^3$ Notice that since x must be above the x-axis, it must be less than the root of $y = 27 - x^2$, which is $3\sqrt{3}$, and must be greater than 0, so we have $0 < x < 3\sqrt{3}$ (if these were less than/equal to and greater than/equal to, you would have to check these endpoints after you find the critical points since they could yield max/min). The maximum can occur when $A'(x) = 0$: $A(x) = 27x - x^3$ $A'(x) = 27 - 3x^2$ Setting $A'(x)$ equal to 0, we get: $27 - 3x^2 = 0 \Rightarrow x^2 = 9 \Rightarrow x = 3$ (not -3 since x must be greater than 0) If we check $x = 3$ in the area function, we get: $A(3) = 27(3) - (3)^3 = 54$ Since there were no other possible points that could yield the maximum (there were the endpoints but they are not included since x cannot be equal to 0 or $3\sqrt{3}$), so the answer is $\boxed{54}$. and $A''(x) = -6x$. The critical points are -3 and 3. But -3 is not in the domain. Since $A''(3) = -18 < 0$ and $A(3\sqrt{3}) = 0$, the maximum occurs at $x = 3$ and so the largest area triangle can ve is $A(3) = 54$. p.241, pr.45

4. Given the curve $y = \frac{x^2+1}{x}$ and derivatives $y' = \frac{x^2-1}{x^2}$ and $y'' = \frac{2}{x^3}$

(a) 5 Points Identify the *domain* of f and any *symmetries* the curve may have.

Solution: The domain of f is $\boxed{(-\infty,0)\cup(0,+\infty) = \mathbb{R} - \{0\}}$. Since $f(-x) = -f(x)$, we note that f is an odd function, so the graph of f is symmetric about the origin. p.241, pr.45

(b) 6 Points Find the intervals where the graph is increasing and decreasing. Find the local maximum and minimum values.

Solution: We have $y' = \frac{x^2-1}{x^2} = 0$ if and only if $x^2 = 1$, that is iff $x = \pm 1$ are the critical points. Note that

$$f' \begin{cases} > 0, & \text{on } (-\infty,-1)\cup(+1,+\infty) \quad \text{f is incresing} \\ < 0, & \text{on } (-1,0)\cup(0,+1) \quad \text{f is decresing} \end{cases}$$

Thus, f is increasing on $(-\infty,-1)\cup(+1,+\infty)$ and decreasing on $(-1,0)\cup(0,+1)$. There are two local extrema one is the local maximum located at $x = -1$ and the other is the local minimum located at $x = 1$, the values are $f(-1) = -2$, $f(1) = 2$. respectively. p.241, pr.45

(c) 6 Points Determine where the graph is concave up and concave down, and find any inflection points.

Solution: We have $y'' = \frac{2}{x^3}$ and so

$$f'' \begin{cases} > 0, & \text{on } (0,+\infty) \quad \text{f is concave up} \\ < 0, & \text{on } (-\infty,0) \quad \text{f is concave down} \end{cases}$$

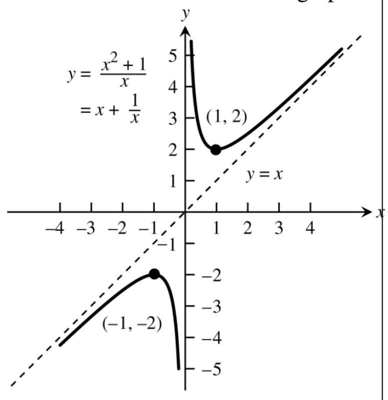
Hence f is concave up on $(0,+\infty)$ and concave down on $(-\infty,0)$. Although f'' changes the sign at $x = 0$, the graph has *no point of inflection* as there is no tangent line at $x = 0$. p.241, pr.45

5. $y = \frac{x^2+1}{x}$ (continued).

(a) 6 Points Find the asymptotes.

Solution: We have $\lim\limits_{x\to 0^+} \dfrac{x^2+1}{x} = +\infty$ and $\lim\limits_{x\to 0^-} \dfrac{x^2+1}{x} = -\infty$. From these we see that the graph has a *vertical asymptote at* $x = 0$. Next *there are no horizontal asymptotes* as $\lim\limits_{x\to\pm\infty} \dfrac{x^2+1}{x} = \pm\infty$ do not exist. For the oblique asymptote, note that $\dfrac{x^2+1}{x} = x + \dfrac{1}{x}$ and we have $\dfrac{1}{x} \to 0$ as $x \to \pm\infty$. This shows that the line $y = x$ *is an oblique asymptote*. Note that $f(x) > x$ if $x > 0$ and $f(x) < x$ if $x < 0$. p.241, pr.45

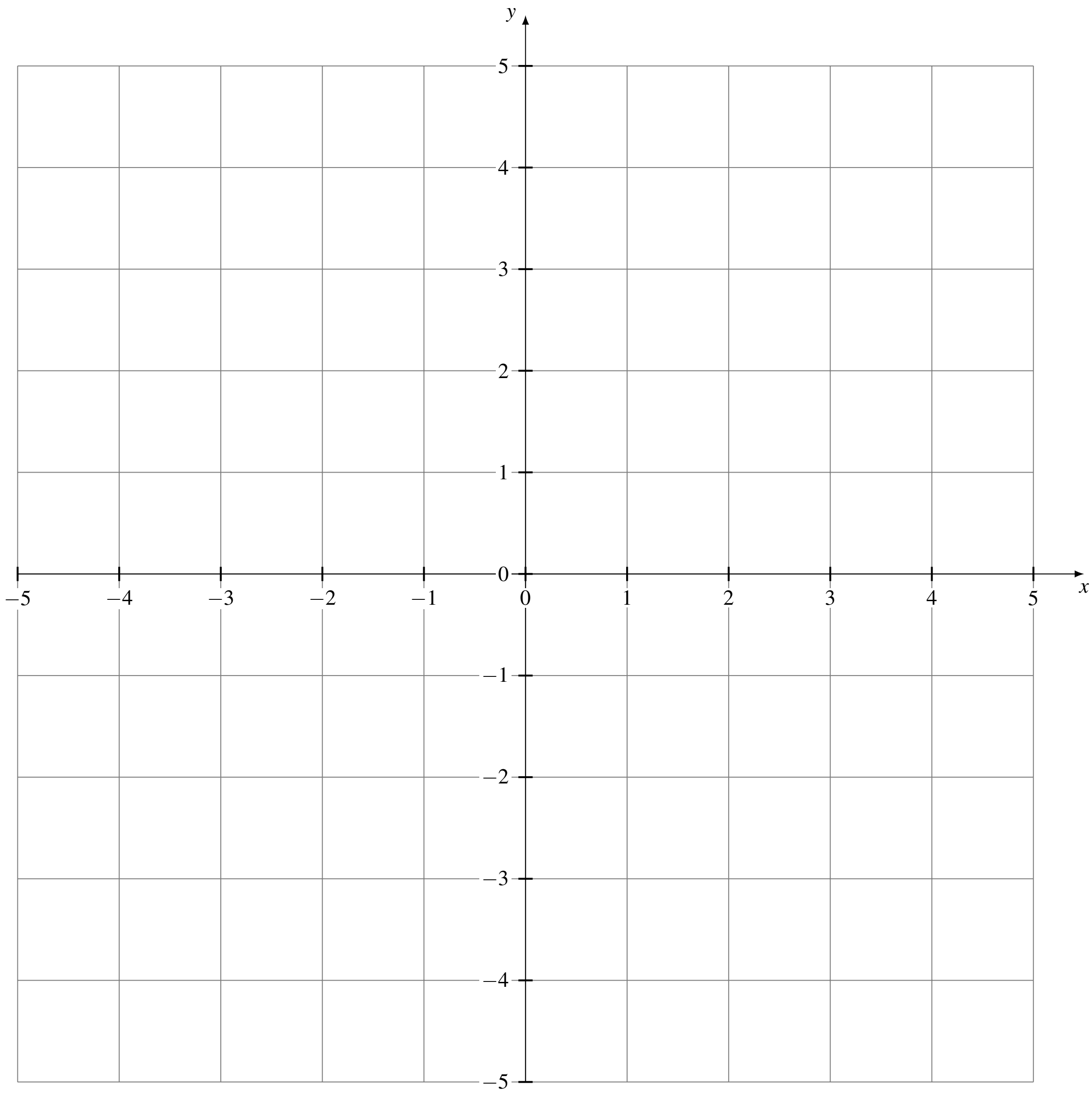
Solution: Here is the graph.



p.241, pr.45

(b) 10 Points Draw the graph of f showing all significant features.

Desired Output



6. 11 Points $\displaystyle\int \left(\sqrt{x} + \sqrt[3]{x}\right)\, dx = ?$

Solution:

$$\int \left(\sqrt{x}+\sqrt[3]{x}\right)\,dx = \int \left(x^{1/2}+x^{1/3}\right)\,dx = \left[\frac{x^{1/2+1}}{1/2+1}+\frac{x^{1/3+1}}{1/3+1}\right]+C = \left[\frac{x^{3/2}}{3/2}+\frac{x^{4/3}}{4/3}\right]+C = \boxed{\frac{2}{3}x^{2/3}+\frac{3}{4}x^{4/3}+C}$$

p.236, pr.27

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7. Suppose

$$g(x) = \begin{cases} x^3, & -2 \le x \le 0 \\ x^2, & 0 < x \le 2. \end{cases}$$

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(a) 5 Points Find the left-hand and right-hand derivatives at $x = 0$.

Solution: The right-hand derivative is $g'_+(0) = \lim_{h\to 0^+} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0^+} \frac{h^2-0^3}{h} = \lim_{h\to 0^+}(h) = 0$. Similarly, the left-hand derivative is $g'_-(0) = \lim_{h\to 0^-} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0^-} \frac{h^3-0^3}{h} = \lim_{h\to 0^-}(h^2) = 0$. Therefore g is differentiable at $x = 0$ and its derivative is $\boxed{g'(0) = 0}$. p.196, pr.6

(b) 3 Points Does g satisfy the hypotheses of the Mean Value Theorem in this interval? Explain.

Solution: We begin with restating MVT. **The Mean Value Theorem:** *Suppose $y = g(x)$ is continuous on a closed interval $[a,b]$ and differentiable on the interval's interior (a,b). Then there is at least one point $c \in (a,b)$ at which* $\frac{g(b)-g(a)}{b-a} = g'(c)$.
Here we have $a = -2$, $b = 2$. First note that $\lim_{x\to -2^+} g(x) = \lim_{x\to -2^+}(x^3) = (-2)^3 = -8 = g(-2)$ and so $g(x)$ is (right-)continuous at $a = -2$ and since $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-}(x^2) = (2)^2 = 4 = g(2)$ so that $g(x)$ is (left-)continuous at $x = 2$. By the solution of part (a), $g(x)$ is differentiable at $x = 0$, so is continuous there. Hence $g(x)$ is continuous on $[-2,2]$. Since x^2 and x^3 are differentiable functions and $g(x)$ is differentiable at $x = 0$, it follows that $g(x)$ is differentiable on $(-2,2)$ and so $g(x)$ satisfies the hypotheses of MVT. p.196, pr.6

(c) 6 Points Find the value(s) of c that satisfy the equation $\dfrac{g(b)-g(a)}{b-a} = g'(c)$ in the conclusion of the Mean Value Theorem for g.

Solution: By part (a), there exists at least one such c. To find all c's, first note that $\dfrac{g(b)-g(a)}{b-a} = g'(c) \Rightarrow 3 = g'(c)$. If $-2 \le x < 0$, then $g'(x) = 3x^2 = 3 \Rightarrow c = \pm 1$. But $c = 1 \notin (-2,0)$ so $\boxed{c=-1}$ is the only solution in this case. Now if $x \in (0,2)$, then $g'(x) = 2x \Rightarrow 3 = g'(c) \Rightarrow 2c = 3 \Rightarrow \boxed{c = \frac{3}{2} \in (0,2)}$. Further $c \neq 0$ as $g'(0) = 0 \neq 3$. Consequently c satisfies the required condition iff $\boxed{c \in \{-1, \frac{3}{2}\}}$. p.196, pr.6