Math 113 Summer 2016 **Second Exam** July 26, 2016

Your Name / Ad - Soyad (**75 min.**) Signature / İmza

Student ID # / Öğrenci No (**mavi tükenmez!**)

Problem	1	2	3	4	Total
Points:	25	20	25	30	100
Score:					

You have **75 minutes**. **(Cell phones off and away!)**. No books, notes or calculators are permitted. **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

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1. (a) (8 Points) Find the *derivative* $f(\theta) = \left(\frac{\sin\theta}{1+\cos\theta}\right)^2$.

Solution: We have

$$\begin{aligned} f'(\theta) &= 2\left(\frac{\sin\theta}{1+\cos\theta}\right)^{2-1}\frac{d}{d\theta}\left(\frac{\sin\theta}{1+\cos\theta}\right) \\ &= 2\frac{\sin\theta}{1+\cos\theta}\frac{(1+\cos\theta)\frac{d}{d\theta}(\sin\theta)-(\sin\theta)\frac{d}{d\theta}(1+\cos\theta)}{(1+\cos\theta)^2} \\ &= 2\frac{\sin\theta}{1+\cos\theta}\frac{(1+\cos\theta)(\cos\theta)-(\sin\theta)(-\sin\theta)}{(1+\cos\theta)^2} \\ &= 2\frac{\sin\theta}{1+\cos\theta}\frac{\cos\theta+\cos^2\theta+\sin^2\theta}{(1+\cos\theta)^2} = 2\frac{\sin\theta}{\cancel{1+\cos\theta}}\frac{\cancel{\cos\theta+1}}{(1+\cos\theta)^2} \\ &= \boxed{\frac{2\sin\theta}{(1+\cos\theta)^2}}. \end{aligned}$$

You must differentiate with respect to θ.

p.147, pr.35

(b) (7 Points) Find the *derivative* $p = \frac{q^2+3}{(q-1)^3+(q+1)^3}$.

Solution: We have, by the quotient rule,

$$\begin{aligned} \frac{dp}{dq} &= \frac{((q-1)^3+(q+1)^3)\frac{d}{dq}(q^2+3)-(q^2+3)\frac{d}{dq}((q-1)^3+(q+1)^3)}{((q-1)^3+(q+1)^3)^2} \\ &= \frac{((q-1)^3+(q+1)^3)(2q)-(q^2+3)(3(q-1)^2+3(q+1)^2)}{((q-1)^3+(q+1)^3)^2} \end{aligned}$$

You must differentiate with respect to p.

p.122 pr.40

(c) (10 Points) Find a function $s = s(t)$ with the properties

$$\frac{d^2s}{dt^2} = \frac{3t}{8}, \quad \left.\frac{ds}{dt}\right|_{t=4} = 3, \quad s(4) = 4$$

.

Solution: We want to find a function $s = s(t)$ with the properties $s''(t) = \frac{4t}{8}$, $s'(4) = 3$, and $s(4) = 4$. Now

$$s''(t) = \frac{3t}{8} \Rightarrow s'(t) = \frac{3t^2}{16} + C \text{ for some constant } C$$

but $s'(4) = 3 \Rightarrow \frac{3\cancel{(4)^2}}{\cancel{16}} + C = 3 \Rightarrow 3 + C = 3 \Rightarrow C = 0. \Rightarrow s'(t) = \frac{3t^2}{16}. \Rightarrow s(t) = \frac{t^3}{16} + D$ for some constant D

again $s(4) = 4 \Rightarrow \frac{4^3}{16} + D = 4 \Rightarrow D = 0 \Rightarrow \boxed{s(t) = t^3/16}$

p.238, pr.84

2. (a) (10 Points) Verify that the point $(-1,0)$ is *on the curve* $6x^2+3xy+2y^2+17y-6=0$ and find the *equation* of the line that is *normal* to the curve at this point.

Solution: If we plug $x=-1$ and $y=0$ simultaneously in the given equation, we get

$$6(-1)^2+3(-1)(0)+2(0)^2+17(0)-6=6-6=0$$

so equation is satisfied which implies in turn that this point lies on the curve. Next we implicitly differentiate this equation, we get

$$\begin{aligned}\frac{d}{dx}(6x^2+3xy+2y^2+17y-6)&=\frac{d}{dx}(0)\\ 12x+3y+3x\frac{dy}{dx}+4y\frac{dy}{dx}+17\frac{dy}{dx}&=0\\ (3x+4y+17)\frac{dy}{dx}&=-12x-3y\\ \frac{dy}{dx}&=\frac{-12x-3y}{3x+4y+17}\\ \left.\frac{dy}{dx}\right|_{(-1,0)}=\frac{-12(-1)-3(0)}{3(-1)+4(0)+17}&=\frac{12}{14}=\frac{6}{7}.\end{aligned}$$

Now the slope of curve at this point is $6/7$. Therefore the normal line must have slope $-7/6$ and equation

$$y-0=-\frac{7}{6}(x+1)\Rightarrow \boxed{7x+6y=-7.}$$

p.154, pr.33(b)

(b) (10 Points) Show that the *linearization* of $f(x)=(1+x)^k$ at $x=0$ is $L(x)=1+kx$.

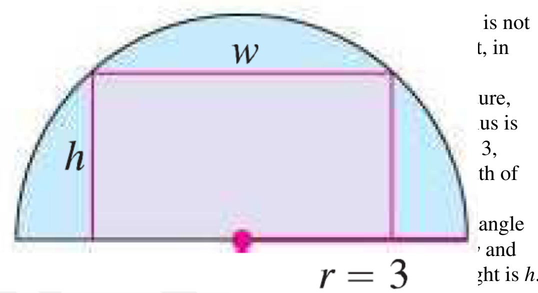
Solution: First $f'(x)=k(1+x)^{k-1}$. So $f(0)=1$ and $f'(0)=k(1+0)^{k-1}=k$. Therefore

$$L(x)=f(0)+f'(0)(x-0)\Rightarrow \boxed{L(x)=1+kx.}$$

p.173, pr.13

3. (a) (12 Points) Determine the dimensions of the rectangle of *largest area* that can be inscribed in a semicircle of radius 3.

Solution: Let h be the length of the side of the rectangle which is perpendicular to the straight-edge of the semicircle, and w be the length of the side which lies along the straight-edge. Draw a couple of radii of the circle and observe that $w/2 = \sqrt{3^2 - h^2}$, so $w = 2\sqrt{9-h^2}$. Thus, the area of the rectangle we wish to maximize is given by



$$A = hw = h = 2h\sqrt{9-h^2} = 2\sqrt{9h^2 - h^4}$$

$$\Rightarrow \frac{dA}{dh} = \frac{2(18h - 4h^3)}{\sqrt{9h^2 - h^4}} = \frac{4h(9-2h^2)}{h\sqrt{9-h^2}} = \frac{4(9-2h^2)}{\sqrt{9-h^2}}$$

Thus, A has a critical point when $9 - 2h^2 = 0$, or when $h^2 = 9/2$ which is equivalent to $h = 3/\sqrt{2}$. Again, verify that this is indeed a maximum (using either the first or second derivative test), and then we see that the dimensions of the rectangle maximizing area are $h = 3/\sqrt{2}$ and

$$w == 2\sqrt{9 - (3/\sqrt{2})^2} = 3\sqrt{2}.$$

The function A is continuous on the closed interval $0 \le h \le 3$ and so has an absolute maximum and an absolute minimum on this interval.

$$A(0) = 0 \quad (\text{ ABS MIN.})$$
$$A(3) = 0 \quad (\text{ ABS MIN.})$$
$$A(3/\sqrt{2}) = (3\sqrt{2})(3/\sqrt{2}) = \boxed{9} \quad (\text{ ABS MAX.})$$

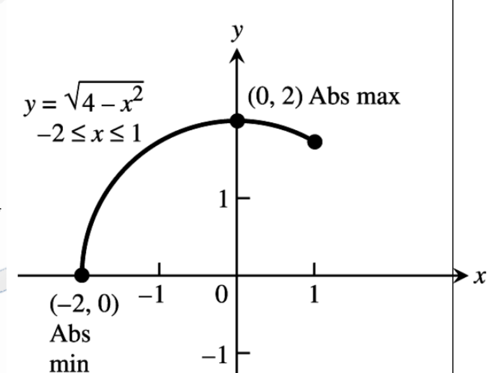
p.222, pr.34

(b) (13 Points) Find the *absolute maximum* and *minimum* values of $g(x) = \sqrt{4-x^2}$ on the interval $-2 \le x \le 1$.

Solution: First the derivative is

$$g'(x) = \frac{d}{dx}\left(4-x^2\right)^{1/2} = \frac{1}{2}\left(4-x^2\right)^{-1/2}(-2x)$$
$$= \frac{-x}{\sqrt{4-x^2}}$$

Critical points are at $x = 0 \in [-2, 1]$. Now we calculate the values of g at the critical point and the endpoints.



$$g(-2) = 0 \textbf{ ABS MIN}$$
$$g(0) = 2 \textbf{ ABS MAX}$$
$$g(1) = \sqrt{3}$$

p.190, pr.29

4. Consider the function $y = \dfrac{x}{x^2-1}$. You may assume that $y' = -\dfrac{x^2+1}{(x^2-1)^2}$ and $y'' = \dfrac{2x^3+6x}{(x^2-1)^3}$. Use this information to graph the function.

   (a) (5 Points) Identify the *domain* of f and any *symmetries* the curve may have.

   Solution: The domain of f is $\boxed{(-\infty,-1)\cup(-1,+\infty)\cup(1,+\infty)=\mathbb{R}-\{\pm 1\}}$. Since $f(-x)=-f(x)$, we note that f is an odd function, so the graph of f is symmetric about the origin. p.241, pr.45

   (b) (7 Points) Give the *asymptotes*.

   Solution: We have $\lim_{x\to 1^+}\dfrac{x}{x^2-1}=+\infty$, $\lim_{x\to 1^-}\dfrac{x}{x^2-1}=-\infty$, $\lim_{x\to -1^+}\dfrac{x}{x^2-1}=-\infty$ and $\lim_{x\to -1^-}\dfrac{x}{x^2-1}=+\infty$. From these we see that the graph has *two vertical asymptotes at* $x=1$ *and* $x=-1$. Next *there is one horizontal asymptote* as $\lim_{x\to\pm\infty}\dfrac{x}{x^2-1}=0$. Hence $y=0$ is the only (horizontal) asymptote. p.105, pr.25

   (c) (5 Points) Find the *intervals* where the graph is *increasing* and *decreasing*. Find the *local maximum* and *minimum* values.

   Solution: We have $y' = -\dfrac{x^2+1}{(x^2-1)^2}=0$ if and only if $x^2=-1$, that is iff no points are the critical points. Note that

$$y' \text{ is } \begin{cases} >0, & \text{nowhere} \quad \text{y is incresing} \\ <0, & \text{on } (-\infty,-1)\cup(-1,1)\cup(1,+\infty) \quad \text{y is decresing} \end{cases}$$

   Thus, y is decreasing everywhere on domain. There are NO local extrema. p.241, pr.45

   (d) (5 Points) Determine where the graph is *concave up* and *concave down*, and find any *inflection points*.

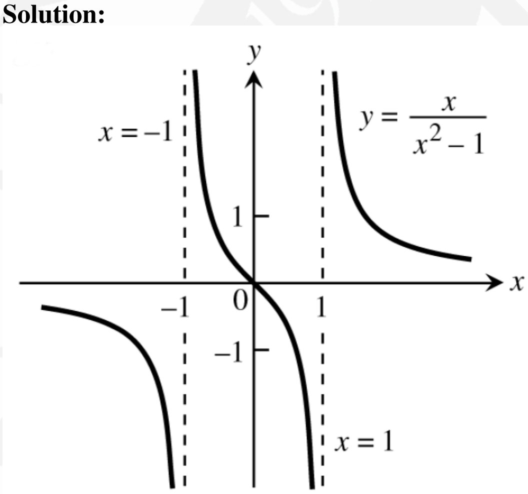
   Solution: We have $y'' = \dfrac{2x^3+6x}{(x^2-1)^3}$ and so

$$y'' \begin{cases} >0, & \text{on } (-1,0)\cup(1,+\infty) \quad \text{y is concave up} \\ <0, & \text{on } (-\infty,-1)\cup(0,1) \quad \text{y is concave down} \end{cases}$$

Hence f is concave up on $(-1,0)\cup(1,+\infty)$ and concave down on $(-\infty,-1)\cup(0,1)$. Also f'' changes the sign at $x=0$, the graph has *one point of inflection* there is also tangent line at $x=0$. p.241, pr.45

(e) (8 Points) *Sketch the graph* of the function. Label the asymptotes, critical points and the inflection points.

Solution:



p.212, pr.89