Your Name

Your Signature

Student ID #

Professor's Name

Your Department

- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 70 min.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	25	
3	35	
4	20	
Total:	100	

1. 20 points Suppose $P = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ and $A = PDP^{-1}$. Compute A^4.

Solution: First we find P^{-1}. Since $\det(P) = (1)(3) - (2)(2) = -1$, we have

$$P^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{1}{\det(P)} \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}.$$

Denoting by I_2 the 2×2 identity matrix, we have

$$A^4 = (PDP^{-1})^4 = (PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1}) = PD\underbrace{(PP^{-1})}_{I_2}D\underbrace{(PP^{-1})}_{I_2}D\underbrace{(PP^{-1})}_{I_2}DP^{-1}$$

$$= PDDDDP^{-1} = PD^4P^{-1}$$

Hence

$$A = PD^4P^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^4 \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1^4 & 0 \\ 0 & 3^4 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 81 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 162 & -81 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 321 & -160 \\ 480 & -239 \end{bmatrix}}$$

2. Let $\mathscr{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be bases for a vector space V, and suppose

$$\mathbf{a}_1 = 4\mathbf{b}_1 - \mathbf{b}_2, \quad \mathbf{a}_2 = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3, \text{ and } \mathbf{a}_3 = \mathbf{b}_2 - 2\mathbf{b}_3.$$

(a) [10 points] Find the change-of-coordinate matrix from $\mathscr{A}$ to $\mathscr{B}$.

Solution:

$$\underset{\mathscr{A} \leftarrow \mathscr{B}}{\mathscr{P}} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

(b) [15 points] Find the coordinate vector $[\mathbf{x}]_{\mathscr{B}}$ relative to the basis $\mathscr{B}$ for $\mathbf{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3$.

Solution: It is possible to solve this in two ways. First, write the vector as a linear combination of vectors in $\mathscr{B}$. That is,

$$\begin{aligned} \mathbf{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3 &= 3(4\mathbf{b}_1 - \mathbf{b}_2) + 4(-\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3) + (\mathbf{b}_2 - 2\mathbf{b}_3) \\ &= ((3)(4) + (4)(-1) + 0)\mathbf{b}_1 + ((3)(-1) + (4)(1) + (1)(1))\mathbf{b}_2 + ((3)(0) + (4)(1) + (1)(-2))\mathbf{b}_3 \\ &= 8\mathbf{b}_1 + 2\mathbf{b}_2 + 2\mathbf{b}_3. \end{aligned}$$

Hence $[\mathbf{x}]_{\mathscr{B}} = \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$.

The second way is directly to use the matrix we found in part (a).

$$\begin{aligned} [\mathbf{x}]_{\mathscr{B}} &= \underset{\mathscr{A} \leftarrow \mathscr{B}}{\mathscr{P}} [\mathbf{x}]_{\mathscr{A}} \\ &= \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix} \end{aligned}$$

3. (a) 20 points Is $\lambda = 1$ an eigenvalue of $A = \begin{bmatrix} 4 & -2 & 3 \\ 0 & -1 & 3 \\ -1 & 2 & -2 \end{bmatrix}$? If so, find one corresponding eigenvector.

Solution: Denote by $p_A(\lambda)$ the characteristic polynomial of A where $p_A(\lambda) = \det(A - \lambda I)$. At $\lambda = 1$, this polynomial takes the value

$$p_A(1) = \det(A - I) = \det \begin{bmatrix} 4-1 & -2 & 3 \\ 0 & -1-1 & 3 \\ -1 & 2 & -2-1 \end{bmatrix} = \begin{vmatrix} 3 & -2 & 3 \\ 0 & -2 & 3 \\ -1 & 2 & -3 \end{vmatrix} = 0$$

which follows from the third column is a scalar multiple of the second. Hence $\lambda = 1$ is an eigenvalue. Next suppose $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector of A corresponding to $\lambda = 1$. Then $\mathbf{x} \in N(A - I)$. Therefore $(A - I)\mathbf{x} = \mathbf{0}$ and and row reduce the augmented matrix for $A - I$.

$$[A - I|0] = \left[\begin{array}{ccc|c} 3 & -2 & 3 & 0 \\ 0 & -2 & 3 & 0 \\ -1 & 2 & -3 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 4 & -6 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then the eigenvector must be of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ (3/2)t \\ t \end{bmatrix} = \frac{t}{2} \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}.$$

Consequently one corresponding eigenvector is $\boxed{\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}}$.

(b) 15 points Show that if A is both diagonalizable and invertible, then so is A^{-1}.

Solution: Since A is diagonalizable, there exists an invertible matrix P and a diagonal matrix D such that

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \Rightarrow A = PDP^{-1}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of A not all of them need be distinct and not all of them need be real.

Since A is invertible, $\det(A) \neq 0$. We know that $\det(A) = \det(PDP^{-1}) = \det(P)\det(D)\det(P^{-1}) = \cancel{\det(P)}\det(D)\frac{1}{\cancel{\det(P)}} = \det(D)$ and that $\det(D) = \lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_n$, the product of all diagonal entries. So $\det(D) \neq 0$, D is also invertible and we have $\lambda_k \neq 0$ for each $k \geq 1$. Hence D^{-1} is given by

$$D^{-1} = \begin{bmatrix} 1/\lambda_1 & 0 & \cdots & 0 \\ 0 & 1/\lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1/\lambda_n \end{bmatrix}$$

Since

$$A^{-1} = \left(P^{-1}\right)^{-1} D^{-1} P^{-1} = PD^{-1}P^{-1}$$

and A^{-1} is also diagonalizable with the same diagonalizing matrix P, and the diagonal matrix is made up of the inverses of the eigenvalues of A.

4. 20 points Find a basis for the eigenspace for $A = \begin{bmatrix} 4 & 0 & -1 \\ 3 & 0 & 3 \\ 2 & -2 & 5 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 3$.

Solution: Suppose $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector of A corresponding to $\lambda = 3$. Then $\mathbf{x} \in N(A-3I)$. Therefore $(A-3I)\mathbf{x} = \mathbf{0}$ and and row reduce the augmented matrix for $A - 3I$.

$$[A-3I|0] = \left[\begin{array}{ccc|c} 4-3 & 0 & -1 & 0 \\ 3 & 0-3 & 3 & 0 \\ 2 & -2 & 5-3 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 3 & -3 & 3 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -3 & 6 & 0 \\ 0 & -2 & 4 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This shows the system has free variable. The general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Consequently a basis for the eigenspace $N(A-3I)$ is $\mathscr{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$.