Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

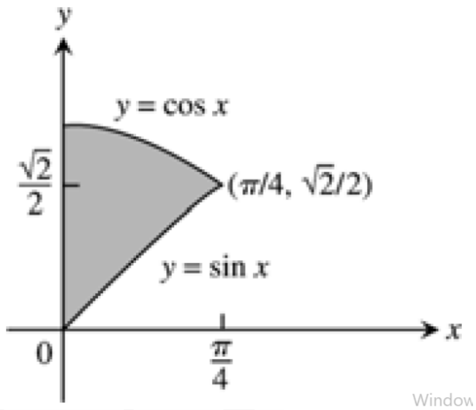
Your Department / Bölüm

- Calculators, cell phones off and away!.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 90 min.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	32	
3	14	
4	14	
5	20	
Total:	100	

1. (a) 10 Points If $R = \{(x,y) \in \mathbb{R}^2 \mid \sin x \le y \le \cos x, \quad 0 \le x \le \pi/4\}$, then use a double integral $\iint_R \mathrm{d}A$ to find the area of R.



Solution:

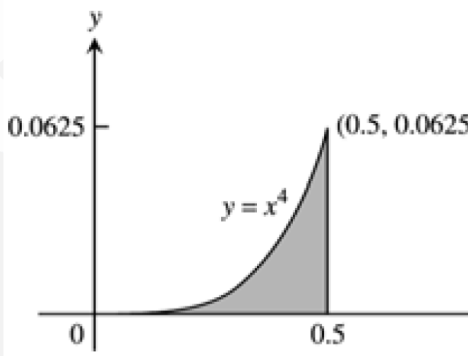
$$\begin{aligned}\text{AREA} &= \int_0^{\pi/4}\int_{\sin x}^{\cos x} \mathrm{d}y\,\mathrm{d}x = \int_0^{\pi/4} (\cos x - \sin x)\,\mathrm{d}x \\ &= [\sin x + \cos x]_0^{\pi/4} \\ &= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1) = \sqrt{2} - 1\end{aligned}$$

p.573, pr.38

(b) 10 Points Reverse the order of integration of the double integral

$$\int_0^{1/16}\int_{y^{1/4}}^{1/2} \cos(16\pi x^5)\,\mathrm{d}x\,\mathrm{d}y$$

and evaluate it.



Solution:

$$\begin{aligned}\int_0^{1/16}\int_{y^{1/4}}^{1/2} \cos(16\pi x^5)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^{1/2}\int_0^{x^4} \cos(16\pi x^5)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^{1/2} x^4 \cos(16\pi x^5)\,\mathrm{d}x = \left[\frac{\sin(16\pi x^5)}{80\pi}\right] \\ &= \boxed{\frac{1}{80\pi}}\end{aligned}$$

p.573, pr.18

2. (a) [11 Points] Find three numbers whose sum is 9 and whose sum of squares is a minimum.

Solution:

$$s(x,y,z)=x^2+y^2+z^2; x+y+z=9 \Rightarrow z=9-x-y \Rightarrow s(x,y)=x^2+y^2+(9-x-y)^2$$

$$\Rightarrow s_x=2x-2(9-x-y)=0 \text{ and } s_y=2y-2(9-x-y)=0$$

$$\Rightarrow \text{ critical point is } (3,3);$$

$$\Rightarrow s_{xx}(3,3)=4, s_{yy}(3,3)=4, s_{xy}(3,3)=2,$$

$$\Rightarrow s_{xx}s_{yy}-s_{xy}^2=12>0 \text{ and } s_{xx}>0$$

$$\Rightarrow \text{ local minimum of } s(3,3,3)=\boxed{27}$$

Therefore the three such numbers are $x=3$, $y=3$, and $z=3$.

p.858, pr.53

(b) [11 Points] Find parametric equations for the line tangent to the curve of intersection of the surfaces $xyz=1$ and $x^2+2y^2+3z^2=6$ at the point $P_0(1,1,1)$.

Solution:

$$\nabla f=yz\mathbf{i}+xz\mathbf{j}+xy\mathbf{k} \Rightarrow \nabla f(1,1,1)=\mathbf{i}+\mathbf{j}+\mathbf{k};$$

$$\nabla g=2x\mathbf{i}+4y\mathbf{j}+6z\mathbf{k} \Rightarrow \nabla g(1,1,1)=2\mathbf{i}+4\mathbf{j}+6\mathbf{k};$$

$$\Rightarrow \mathbf{v}=\nabla f\times\nabla g=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&1\\2&4&6\end{vmatrix}=2\mathbf{i}-4\mathbf{j}+2\mathbf{k}$$

$$\Rightarrow \text{ Tangent Line: } \boxed{x=1+2t,\quad y=1-4t,\quad z=1+2t}$$

p.192, pr.87

(c) [10 Points] If it exists, find the limit $\displaystyle\lim_{\substack{(x,y)\to(1,1)\\ y\neq x}}\frac{x^2-2xy+y^2}{x-y}$.

Solution:

$$\lim_{\substack{(x,y)\to(1,1)\\ y\neq x}}\frac{x^2-2xy+y^2}{x-y}=\lim_{(x,y)\to(1,1)}\frac{\cancel{(x-y)}(x-y)}{\cancel{x-y}}=\lim_{(x,y)\to(1,1)}(x-y)=(1-1)=\boxed{0}$$

3. [14 Points] Evaluate the integral $\int_0^1 \frac{1}{(x+1)(x^2+1)}\,dx$.

Solution: The appropriate partial fraction decomposition is

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \Rightarrow 1 = A(x^2+1)+(Bx+C)(x+1)$$

$$1 = Ax^2+A+Bx^2+Bx+Cx+C \Rightarrow 1 = (A+B)x^2+(B+C)x+(A+C)$$

$$\Rightarrow A+B=0, B+C=0, A+C=1 \Rightarrow B=-A, C=-B \Rightarrow C=A \Rightarrow A+A=1 \Rightarrow A=\frac{1}{2}, B=-\frac{1}{2}, C=\frac{1}{2};$$

$$\Rightarrow \int_0^1 \frac{1}{(x+1)(x^2+1)}\,dx = \frac{1}{2}\int_0^1 \frac{1}{x+1}\,dx + \frac{1}{2}\int_0^1 \frac{(-x+1)}{x^2+1}\,dx = \left[\frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x\right]_0^1$$

$$= \left(\frac{1}{2}\ln 2 - \frac{1}{4}\ln 2 + \frac{1}{2}\tan^{-1}1\right) - \left(\frac{1}{2}\ln 1 - \frac{1}{4}\ln 1 + \frac{1}{2}\tan^{-1}0\right) = \frac{1}{4}\ln 2 + \frac{1}{2}\left(\frac{\pi}{4}\right) = \boxed{\frac{(\pi+2\ln 2)}{8}}$$

p.822, pr.65

4. [14 Points] Find the radius and interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3x-1)^n}{n^2}$.

Solution: Let $u_n = \frac{(-1)^{n-1}(3x-1)^n}{n^2}$. Then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{\frac{(-1)^n(3x-1)^{n+1}}{(n+1)^2}}{\frac{(-1)^{n-1}(3x-1)^n}{n^2}}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|(3x-1)\frac{n^2}{(n+1)^2}\right| < 1 \Rightarrow |3x-1|\underbrace{\lim_{n\to\infty}\left(\frac{n^2}{(n+1)^2}\right)}_{=1} < 1$$

$$\Rightarrow |3x-1| < 1$$

$$\Rightarrow -1 < 3x-1 < 1 \Rightarrow 0 < 3x < 2 \Rightarrow 0 < x < \frac{2}{3}$$

When $x=0$, we have $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-1)^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^{2n-1}}{n^2} = -\sum_{n=1}^{\infty}\frac{1}{n^2}$, a nonzero constant multiple of a convergent p-series which is absolutely convergent;
When $x=\frac{2}{3}$, we have $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(1)^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$, which converges absolutely. So the radius of convergence is $R=1/3$; the interval of convergence is $0 \le x \le \frac{2}{3}$.

p.687, pr.17(a)

5. (a) [10 Points] Find parametric equations of the line through $Q(0,1,1)$ parallel to the plane $2x-y-z=4$ and orthogonal to $\mathbf{v}=\mathbf{i}+\mathbf{j}+\mathbf{k}$.

Solution: Notice that

$\mathbf{n} = 2\mathbf{i}-\mathbf{j}-\mathbf{k}$ is normal to the plane

$$\Rightarrow \mathbf{n}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 0\mathbf{i}-3\mathbf{j}+3\mathbf{k} = \boxed{-3\mathbf{j}+3\mathbf{k}}$$

is orthogonal to $\mathbf{v}$ and parallel to the plane. The line has parametric equations
$\boxed{x = 0+0t = 0,\ y = 1-3t,\ z = 1+3t\ \infty < t < \infty}$

p.583, pr.17

(b) [10 Points] Find the distance from the point $Q(0,1,1)$ to the plane $2x-y-z=4$.

Solution: We shall use the distance formula $d = \left|\frac{\vec{PQ}\bullet\mathbf{n}}{|\mathbf{n}|}\right|$. Here $P(2,0,0)$ is a point on the plane and $\mathbf{n}=2\mathbf{i}-\mathbf{j}-\mathbf{k}$ is a vector normal to the plane. Therefore,

$$d = \left|\frac{\vec{PQ}\bullet\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{(-2\mathbf{i}+\mathbf{j}+\mathbf{k})\bullet(2\mathbf{i}-\mathbf{j}-\mathbf{k})}{|2\mathbf{i}-\mathbf{j}-\mathbf{k}|}\right| = \left|\frac{-6}{\sqrt{6}}\right| = \boxed{\sqrt{6}}$$

p.583, pr.17