Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- Calculators, cell phones off and away!.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

Do not write in the table to the right.

Problem	Points	Score
1	32	
2	35	
3	33	
Total:	100	

1. (a) 10 Points Use Lagrange Multipliers to find the maximum and minimum values of $f(x,y,z) = x-2y+5z$ on the sphere $x^2+y^2+z^2 = 30$.

Solution: $\nabla f = \mathbf{i}-2\mathbf{j}+5\mathbf{k}$ and $\nabla g = 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g$ implies $\mathbf{i}-2\mathbf{j}+5\mathbf{k} = \lambda(2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k})$.

If we equate the components, we get $1 = 2x\lambda$, $-2 = 2y\lambda$, and $5 = 2z\lambda$. So $x = \frac{1}{2\lambda} \Rightarrow y = -\frac{1}{2\lambda} = -2x$, $z = \frac{5}{2\lambda} = 5x$, $x^2+(-2x)^2+(5x)^2 = 30 \Rightarrow x = \pm 1$.

Thus $x = 1$, $y = -2$, $z = 5$ or $x = -1$, $y = 2$, $z = -5$.

Therefore $f(1,-2,5) = 30$ is the maximum value and $f(-1,2,-5) = -30$ is the minimum value.

p.867, pr.23

(b) 12 Points Find the *directional derivative* of $f(x,y,z) = xy+yz+xz$ at $P_0(1,-1,2)$ in the direction of $\mathbf{v} = 3\mathbf{i}+6\mathbf{j}-2\mathbf{k}$.

Solution:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v} = 3\mathbf{i}+6\mathbf{j}-2\mathbf{k}}{\sqrt{3^2+6^2+(-2)^2}} = \frac{3}{7}\mathbf{i}+\frac{6}{7}\mathbf{j}-\frac{2}{7}\mathbf{k}$$

$$f_x(x,y,z) = y+z \Rightarrow f_x(1,-1,2) = 1$$

$$f_y(x,y,z) = x+z \Rightarrow f_y(1,-1,2) = 3;$$

$$f_z(x,y,z) = y+x \Rightarrow f_z(1,-1,2) = 0$$

$$\nabla f = \mathbf{i}+3\mathbf{j}$$

$$\Rightarrow (D_{\mathbf{u}f})_{P_0} = \nabla f \bullet \mathbf{u} = (\mathbf{i}+3\mathbf{j}) \bullet (\frac{3}{7}\mathbf{i}+\frac{6}{7}\mathbf{j}-\frac{2}{7}\mathbf{k}) = \frac{3}{7}+\frac{18}{7} = \boxed{3}$$

p.317, pr.33

(c) 10 Points Find the gradient and use it to find the equation of the line tangent to the curve $x^2-xy+y^2 = 7$ at the point $P(-1,2)$.

Solution: Let $f(x,y) = x^2-xy+y^2 = 7$. Then

$$f_x(x,y) = 2x-y \Rightarrow f_x(-1,2) = -4$$

$$f_y(x,y) = -x+2y \Rightarrow f_y(-1,2) = 5$$

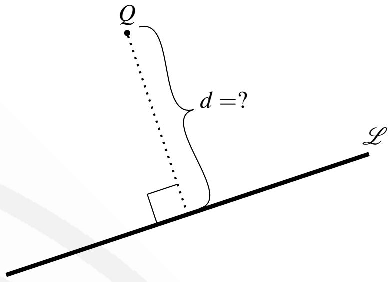
$$\nabla f(-1,2) = -4\mathbf{i}+5\mathbf{j}$$

$\Rightarrow$ **Tangent line** $: -4(x+1)+5(y-2) = 0$

$$\Rightarrow \boxed{-4x+5y-14 = 0}$$

p.840, pr.28

2. (a) 14 Points Find the distance from the point $Q(1,2,1)$ to the line $\mathscr{L}: \begin{cases} x = 2+2t, \\ y = 1+6t, \\ z = 3 \end{cases}$



Solution: We shall use the distance formula $d = \dfrac{|\vec{PQ}\times \mathbf{v}|}{|\mathbf{v}|}$. Here (by letting $t=0$) $P(2,1,3)$ is a point on $\mathscr{L}$ and $\mathbf{v} = 2\mathbf{i}+6\mathbf{j}+0\mathbf{k}$ is a vector that is parallel to $\mathscr{L}$. Now we have $\vec{PQ} = -\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and so

$$\vec{PQ}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -2 \\ 2 & 6 & 0 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 1 & -2 \\ 6 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -1 & -2 \\ 2 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -1 & 1 \\ 2 & 6 \end{vmatrix}$$
$$= 12\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$$

Therefore, we have

$$d = \frac{|\vec{PQ}\times\mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{(12)^2+(-4)^2+(-8)^2}}{\sqrt{(2)^2+(6)^2+(0)^2}} = \frac{\sqrt{224}}{\sqrt{40}} = \boxed{\frac{2\sqrt{7}}{\sqrt{5}}}$$

p.695, pr.37

(b) 11 Points Find the radius and interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$.

Radius of Convergence: __________________. Interval of Convergence: __________________

Solution: Let $u_n = \dfrac{(x-1)^n}{\sqrt{n}}$. Then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{\frac{(x-1)^{n+1}}{\sqrt{n+1}}}{\frac{(x-1)^n}{\sqrt{n}}}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{\sqrt{n+1}}\,\frac{\sqrt{n}}{(x-1)^n}\right| < 1 \Rightarrow |x-1|\sqrt{\lim_{n\to\infty}\frac{n}{n+1}} < 1$$
$$\Rightarrow |x-1| < 1$$
$$\Rightarrow -1 < x-1 < 1$$
$$\Rightarrow 0 < x < 2$$

When $x=0$, we have $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/2}}$, a conditionally convergent series.

When $x=2$, we have $\displaystyle\sum_{n=1}^{\infty}\frac{(2-1)^n}{n^{1/2}} = \sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$,a divergent series. So the radius of convergence is $R=1$; the interval of convergence is $0 \le x < 2$.

p.583, pr.17

(c) 10 Points Investigate the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n!}$.

○ Converges. ○ Diverges. Test Used: __________________

Solution: Use Ratio Test. Let $u_n = \dfrac{n+1}{n!} > 0$. Then

$$\rho = \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\frac{n+2}{(n+1)!}}{\frac{n+1}{n!}} = \lim_{n\to\infty}\left[\frac{n+2}{(n+1)!}\,\frac{n!}{n+1}\right] = \lim_{n\to\infty}\frac{n+2}{(n+1)\cancel{n!}}\,\frac{\cancel{n!}}{n+1} = \lim_{n\to\infty}\frac{n+2}{(n+1)^2}$$
$$= \lim_{n\to\infty}\frac{\frac{1}{n}+\frac{2}{n^2}}{1+\frac{2}{n}+\frac{1}{n^2}} = 0 < 1$$

The series converges.

p.600, pr.19

3. (a) 10 Points Evaluate the improper integral $\int_0^\infty \frac{dx}{x^2+1}$.

Solution: By the definition of Improper Integrals of Type I, we have

$$\int_0^\infty \frac{dx}{x^2+1} = \lim_{b\to\infty}\int_0^b \frac{dx}{x^2+1} = \lim_{b\to\infty}\int_0^b \frac{dx}{x^2+1} = \lim_{b\to\infty}\left[\tan^{-1}x\right]_0^b = \lim_{b\to\infty}\left[\tan^{-1}(b)-\tan^{-1}(0)\right] = \lim_{b\to\infty}(\tan^{-1}(b)) = \frac{\pi}{2}$$

So the improper integral converges and has value $\pi/2$.

p.632, pr.13

(b) 10 Points Express the integrand as a sum of partial fractions and evaluate the integral $\int \frac{1}{(x^2-1)^2}\,dx$.

Solution: The appropriate decomposition is

$$\Rightarrow \frac{1}{(x^2-1)^2} = \frac{1}{((x+1)(x-1))^2} = \frac{1}{(x+1)^2(x-1)^2} = \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-1}+\frac{D}{(x-1)^2}$$

$$\Rightarrow (x+1)^2(x-1)^2\left[\frac{1}{(x+1)^2(x-1)^2}\right] = (x+1)^2(x-1)^2\left[\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-1}+\frac{D}{(x-1)^2}\right]$$

Clear the fractions:

$$1 = A(x+1)(x-1)^2 + B(x-1)^2 + C(x-1)(x+1)^2 + D(x+1)^2.$$

Letting $x=-1$ gives $B=\frac{1}{4}$ and $x=1$ gives $D=\frac{1}{4}$. To find A and C, we put $x=0$ and $x=2$. If we let $x=0$, then we get $1=A+B-C+D$ and for $x=2$, we have $1=3A+B+9C+9D$. So we have the system

$$\begin{cases} A+B-C+D=1 \\ 3A+B+9C+9D=1 \\ B=\frac{1}{4}, D=\frac{1}{4} \end{cases} \Rightarrow \begin{cases} A+\frac{1}{4}-C+\frac{1}{4}=1 \\ 3A+\frac{1}{4}+9C+\frac{9}{4}=1 \end{cases} \Rightarrow \begin{cases} A-C=\frac{1}{2} \\ 3A+9C=-\frac{3}{2} \end{cases} \Rightarrow \begin{cases} A=\frac{1}{4} \\ C=-\frac{1}{4} \end{cases}$$

Then the integral becomes

$$\int \frac{1}{(x^2-1)^2}\,dx = \int\left(\frac{1/4}{x+1}+\frac{1/4}{(x+1)^2}+\frac{-1/4}{x-1}+\frac{1/4}{(x-1)^2}\right)dx$$

$$= \frac{1}{4}\ln|x+1| - \frac{1}{4}\frac{1}{x+1} - \frac{1}{4}\ln|x-1| - \frac{1}{4}\frac{1}{x-1} = \boxed{\frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{1}{2}\frac{x}{x^2-1} + K}$$

p.468, pr.33

(c) 13 Points Evaluate the integral $\int_0^{\ln 2} \tanh(2x)\,dx$.

Solution: Let $u=\cosh(2x)$ and so $du = 2\sinh(2x)\,dx$. When $x=0$, we have $u=\cosh(0)=1$ and when $x=\ln 2$, we have

$$u = \cosh(2\ln 2) = \cosh(\ln 4) = \frac{e^{\ln 4}+e^{-\ln 4}}{2} = \frac{e^{\ln 4}+e^{\ln 4^{-1}}}{2} = \frac{4+\frac{1}{4}}{2} = \frac{17}{8}.$$

Therefore

$$\int_0^{\ln 2}\tanh(2x)\,dx = \int_0^{\ln 2}\frac{\sinh(2x)}{\cosh(2x)}\,dx = \frac{1}{2}\int_0^{\ln 2}\frac{1}{\cosh(2x)}2\sinh(2x)\,dx = \frac{1}{2}\int_1^{\frac{17}{8}}\frac{1}{u}\,du$$

$$= \left[\frac{1}{2}\ln|u|\right]_1^{\frac{17}{8}}$$

$$= \frac{1}{2}\left[\ln\frac{17}{8}-\ln 1\right] = \boxed{\frac{1}{2}\ln\frac{17}{8}}$$

p.481, pr.28