Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- Calculators, cell phones off and away!.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 90 min.

Do not write in the table to the right.

Problem	Points	Score
1	33	
2	35	
3	11	
4	11	
5	10	
Total:	100	

1. (a) [11 Points] Reverse the order of integration of the double integral

$$\int_0^1 \int_y^1 x^2 e^{xy}\, \mathrm{d}x\, \mathrm{d}y$$

and evaluate it.

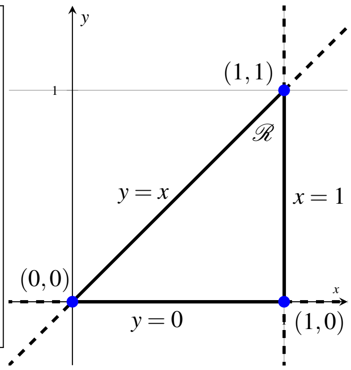
Solution:

$$\int_0^1 \int_y^1 x^2 e^{xy}\, \mathrm{d}x\, \mathrm{d}y = \int_0^1 \int_0^x x^2 e^{xy}\, \mathrm{d}y\, \mathrm{d}x = \int_0^1 \left[x^2 \frac{1}{x} e^{xy}\right]_0^x \mathrm{d}x$$

$$= \int_0^1 \left(x e^{x^2} - x\right) \mathrm{d}x = \left[\frac{1}{2} e^{x^2} - \frac{x^2}{2}\right]_0^1$$

$$= \boxed{\frac{e-2}{2}}$$

p.695, pr.37



(b) [11 Points] If R is the region bounded by $y = x^3$, $x = 0$, and $y = 8$ then write an iterated integral for $\iint_R \mathrm{d}A$ using (i) first with respect to y then with respect to x, (ii) first with respect to x then with respect to y. (do not evaluate the integrals)

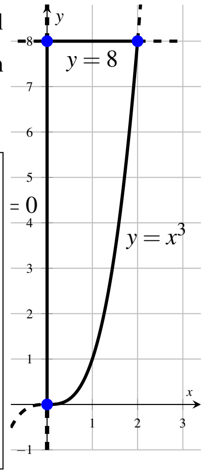
Solution: (i) using vertical cross-sections

$$\iint_R \mathrm{d}A = \int_0^2 \int_{x^3}^8 \mathrm{d}y\, \mathrm{d}x$$

(ii) using horizontal cross-sections

$$\iint_R \mathrm{d}A = \int_0^8 \int_0^{y^{1/3}} \mathrm{d}y\, \mathrm{d}x$$

p.573, pr.38



(c) [11 Points] Find the double intgral of $f(x,y) = x^2 + y^2$ over the triangular region with vertices $(0,0)$, $(1,0)$, $(0,1)$.

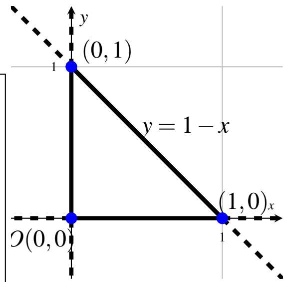
Solution: If we use vertical cross-sections, then we have

$$\int_0^1 \int_0^{1-x} \left(x^2 + y^2\right) \mathrm{d}y\, \mathrm{d}x = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} \mathrm{d}x = \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3}\right] \mathrm{d}x$$

$$= \int_0^1 \left[x^2 - x^3 + \frac{(1-x)^3}{3}\right] \mathrm{d}x$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12}\right]_0^1 = \left(\frac{1}{3} - \frac{1}{4} - 0\right) - \left(0 - 0 - \frac{1}{12}\right) = \boxed{\frac{1}{6}}$$

p.573, pr.18



2. (a) 12 Points Find the points on the surface $z^2 = xy+4$ closest to the origin.

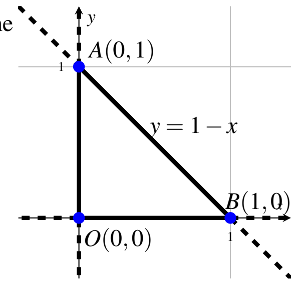
Solution: Let $f(x,y,z) = x^2+y^2+z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$ and $\nabla g = -y\mathbf{i}-x\mathbf{j}+2z\mathbf{k}$ so that $\nabla f = \lambda\nabla g \Rightarrow 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k} = \lambda(-y\mathbf{i}-x\mathbf{j}+2z\mathbf{k}) \Rightarrow 2x = -y\lambda$, $2y = -x\lambda$, and $2z = 2z\lambda \Rightarrow \lambda = 1$ or $z = 0$.

CASE 1: $\lambda = 1 \Rightarrow 2x = -y$ and $2y = -x \Rightarrow y = 0$ and $x = 0 \Rightarrow z^2-4 = 0 \Rightarrow z = \pm 2$ and $x = y = 0$.

CASE 2: $z = 0 \Rightarrow -xy-4 = 0 \Rightarrow y = -\frac{4}{x}$. Then $2x = \frac{4}{x}\lambda \Rightarrow \lambda = \frac{x^2}{2}$, and $-\frac{8}{x} = -\frac{x}{\lambda} \Rightarrow -\frac{8}{x} = -x\left(\frac{x^2}{2}\right) \Rightarrow x^4 = 16 \Rightarrow x = \pm 2$.
Thus $x = 2$ and $y = -2$ or $x = -2$ and $y = 2$.
Therefore we get four points: $(2,-2,0)$, $(-2,2,0)$, $(0,0,2)$. and $(0,0,-2)$. But *the points* $(0,0,2)$ *and* $(0,0,-2)$ *are closest to the origin* since they are 2 units away and the others are $2\sqrt{2}$ units away.

p.491, pr.86

(b) 13 Points Find the absolute maxima and minima of $f(x,y) = 4x-8xy+2y+1$ on the triangular plate bounded by the lines
$x = 0$, $y = 0$, $x+y = 1$.



Solution: Let the vertices be $A(0,1)$, $B(1,0)$, $O(0,0)$.

Along OA, $f(x,y) = f(0,y) = 2y+1$ on $0 \le y \le 1$;
$f'(0,y) = 2 \Rightarrow$ no interior critical points;
$f(0,0) = 1$ and $f(0,1) = 3$.

Along OB, $f(x,y) = f(x,0) = 4x+1$ on $0 \le x \le 1$;
$f'(x,0) = 4 \Rightarrow$ no interior critical points;
$f(1,0) = 5$.

Along AB, $f(x,y) = f(x,-x+1) = 8x^2-6x+3$ on $0 \le x \le 1$;
$f'(x,-x+1) = 16x-6 = 0 \Rightarrow x = \frac{3}{8}$ and $y = \frac{5}{8}$; $f(\frac{3}{8},\frac{5}{8}) = \frac{15}{8}$,
$f(0,1) = 3$ and $f(1,0) = 5$ interior critical points;
$f(1,0) = 5$.

For interior points, $f_x(x,y) = 4-8y = 0$ and $f_y(x,y) = -8x+2 = 0 \Rightarrow y = \frac{1}{2}$
and $x = \frac{1}{4}$ which is an interior critical point with $f(\frac{1}{4},\frac{1}{2}) = 2$. Therefore the absolute maximum is 5 at $(1,0)$ and the absolute minimum is 1 at $(0,0)$.

p.192, pr.87

(c) 10 Points Find equations for the (a) tangent plane and (b) normal line at $P_0(0,1,2)$ on the surface $\cos(\pi x)-x^2y+e^{xz}+yz = 4$.

(i) tangent line equation:

Solution: First the gradient vector is $\nabla f = (-\pi\sin(\pi x)-2xy+ze^{xz})\mathbf{i}+(-x^2+z)\mathbf{j}+(xe^{xz}+y)\mathbf{k}$. Then $\nabla f(0,1,2) = 2\mathbf{i}+2\mathbf{j}+\mathbf{k}$

Tangent Plane: $2(x-0)+2(y-1)+1(z-2) = 0 \Rightarrow$ $\boxed{2x+2y+z = 4}$

(ii) normal line equation:

Solution:

$$\text{normal line: } \begin{cases} x = 2t \\ y = 1+2t \\ z = 2+4 \end{cases}$$

3. 11 Points Evaluate the integral $\int \frac{x^2}{4+x^2}\,\mathrm{d}x$.

Solution: The integrand is a rational function where the degree of numerator is not strictly less than that of denominator. By long division,

$$\int \frac{x^2}{4+x^2}\,\mathrm{d}x = \int \frac{4+x^2-4}{4+x^2}\,\mathrm{d}x = \int \left(\frac{4+x^2}{4+x^2} - \frac{4}{4+x^2}\right)\mathrm{d}x = \int \left(1 - \frac{4}{4}\frac{1}{1+(\frac{x}{2})^2}\right)\mathrm{d}x = 2\int \left(1 - \frac{1}{1+(\frac{x}{2})^2}\right)\frac{1}{2}\,\mathrm{d}x$$

$$= 2\int \left(1 - \frac{1}{1+u^2}\right)\mathrm{d}u$$

$$= 2\left(u - \tan^{-1} u\right) + c = 2\left(\frac{x}{2} - \tan^{-1}\frac{x}{2}\right) + c = \boxed{x - 2\tan^{-1}\left(\frac{x}{2}\right) + c}$$

p.822, pr.65

4. 11 Points Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$ converge absolutely or conditionally or diverge? Give reasons.

Solution: This is an alternating series of the form $\sum_{n=1}^{\infty}(-1)^n a_n$ with $a_n = \frac{1}{1+\sqrt{n}} > 0$ for all $n \geq 1$. Using the Alternating Series Test (AST),

- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{1+\sqrt{n}} = 0$

and

- $\frac{a_{n+1}}{a_n} = \frac{1}{1+\sqrt{n+1}} \cdot \frac{1+\sqrt{n}}{1} = \frac{1+\sqrt{n}}{1+\sqrt{n+1}} < 1$ for all $n \geq 1$,

so $a_{n+1} < a_n$ for all $n \geq 1$, so the series converges.
But by the Limit Comparison Test (LCT), letting

$$a_n = \frac{1}{1+\sqrt{n}}, \quad b_n = \frac{1}{\sqrt{n}}$$

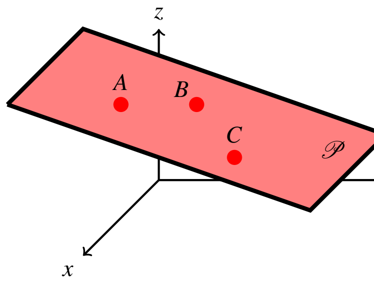
we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt{n}}+1} = 1$$

so $0 < c = 1 < \infty$ and $\sum \frac{1}{\sqrt{n}}$ diverges implies $\sum \frac{1}{1+\sqrt{n}}$ diverges too.
Therefore $\sum_{n=1}^{\infty} |(-1)^n a_n| = \sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ diverges. So $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$ *converges conditionally*.

p.687, pr.17(a)

5. 10 Points Find an equation for the plane through $A(1,1,-1)$, $B(2,0,2)$, $C(0,-2,1)$.



Solution: First we find a normal vector to the plane:

$$\vec{AB} = (2-1)\mathbf{i} + (0-1)\mathbf{j} + (2-(-1))\mathbf{k}$$
$$= \mathbf{i} - \mathbf{j} + 3\mathbf{k}$$
$$\vec{AC} = (0-1)\mathbf{i} + (-2-1)\mathbf{j} + (1-(-1))\mathbf{k}$$
$$= -\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$$
$$\Rightarrow \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix}$$
$$= 7\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$$

is normal to the plane

$$\Rightarrow 7(x-2) - 5(y-0) - 4(z-2) = 0$$

hence $\boxed{7x - 5y - 4z = 6}$ is the equation of the plane.

p.583, pr.17