Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- Calculators, cell phones off and away!.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

Do not write in the table to the right.

Problem	Points	Score
1	32	
2	35	
3	33	
Total:	100	

1. (a) 10 Points Use Lagrange Multipliers to find the *minimum distance* from the surface $x^2 - y^2 - z^2 = 1$ to the origin (the point $O(0,0,0)$).

Solution: Let $f(x,y,z) = x^2 + y^2 + z^2$ be the square of the distance from the origin.

Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = 2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g$ implies $2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k})$.

If we equate the components, we get $2x = 2x\lambda$, $2y = -2y\lambda$, and $2z = -2z\lambda$. So $x = 0$ or $\lambda = 1$.

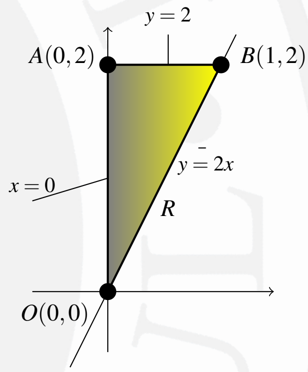
CASE I: $\lambda = 1 \Rightarrow 2y = -2y \Rightarrow y = 0$; $2z = -2z \Rightarrow z = 0$; $\Rightarrow x^2 - 1 = 0$; $\Rightarrow x = \pm 1$; and $y = z = 0$

CASE II: $x = 0 \Rightarrow -y^2 - z^2 = 1$ which has no solution.

Therefore the points $(\pm 1, 0, 0)$ on the surface $x^2 - y^2 - z^2 = 1$ are closest to the origin. The minimum distance is $\boxed{1}$.

p.726, pr.5(d)

(b) 12 Points Find the *absolute maximum and minimum* values of $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate R bounded by the lines $x = 0$, $y = 2$, $y = 2x$.



Solution:

• *On OA*, we have $f(x,y) = f(0,y) = y^2 - 4y + 1$ for $0 \le y \le 2$. So $f'(0,y) = 2y - 4 = 0 \Rightarrow y = 2$ and $x = 0 \Rightarrow (0,2)$ So $f(0,0) = 1$ and $f(0,2) = -3$.

• *On AB*, we have $f(x,y) = f(x,2) = 2x^2 - 4x - 3$ for $0 \le x \le 1$. So $f'(x,2) = 4x - 4 = 0 \Rightarrow x = 1$. $f(0,2) = -3$ and $f(1,2) = -5$.

• *On OB*, we have $f(x,y) = f(x,2x) = 6x^2 - 12x + 1$ for $0 \le x \le 1$; endpoint values have been found above; $f'(x,2x) = 12x - 12 = 0 \Rightarrow x = 1$ and $y = 2 \Rightarrow (1,2)$ is not an interior point of OB.

• *Interior Points of this triangular region R*: $f_x(x,y) = 4x - 4 = 0 \Rightarrow x = 1$ and $f_y(x,y) = 2y - 4 = 0 \Rightarrow y = 2 \Rightarrow (1,2)$ is not an interior point of R.

• Therefore the absolute maximum is 1 at $(0,0)$ and the absolute minimum is -5 at $(1,2)$,

p.317, pr.33

(c) 10 Points Find an equation of the plane that is tangent to the surface $z = e^{-(x^2+y^2)}$ at the point $(0,0,1)$.

Solution: $f_x(x,y) = -2xe^{-(x^2+y^2)}$ and $f_y(x,y) = -2ye^{-(x^2+y^2)}$. So $f_x(0,0) = 0$ and $f_y(0,0) = 0$. A normal vector is $\mathbf{n} = \langle 0,0,-1 \rangle$ and the tangent plane has equation $0(x-0) + 0(y-0) - (z-1) = 0 \Rightarrow \boxed{z = 1}$

p.847, pr.10

2. (a) 14 Points Find the point of intersection for the lines and find a vector orthogonal to both lines. $\begin{cases} L1: x=-1+t,\ y=2+t,\ z=1-t & -\infty < t \\ L2: x=1-4s,\ y=1+2s,\ z=2-2s & -\infty \end{cases}$

Solution: To find the point of intersection we equate the x- and y- coordinates of these lines.

$$\begin{cases} x=-1+t=1-4s \\ y=1+2s=2+t \end{cases} \Rightarrow \begin{cases} 4s+t=2 \\ -2s+t=-1 \end{cases} \Rightarrow \begin{cases} s=1/2 \\ t=0 \end{cases} \Rightarrow \begin{cases} x=-1 \\ y=2 \\ z=1 \end{cases}.$$

Therefore $\boxed{L1\cap L2=\{(-1,2,1)\}}$.

Next the cross product of $\mathbf{v_1}=\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{v_2}=-4\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ has the same direction as the normal to both lines.

$$\begin{aligned} \mathbf{n} &= \vec{PQ}\times\vec{PR}=\langle 1,1,-1\rangle\times\langle -4,2,-2\rangle \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ -4 & 2 & -2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix}\mathbf{i}-\begin{vmatrix} 1 & -1 \\ -4 & -2 \end{vmatrix}\mathbf{j}+\begin{vmatrix} 1 & 1 \\ -4 & 2 \end{vmatrix}\mathbf{k} \\ &= \boxed{6\mathbf{j}+6\mathbf{k}} \end{aligned}$$

p.741, pr.29

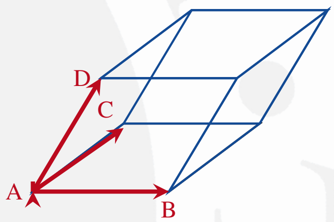
(b) 11 Points Find the volume of the parallelepiped if four of its vertices are $A(0,0,0)$, $B(1,2,0)$, $C(0,-3,2)$, and $D(3,-4,5)$.

Solution: $\vec{AB}=\mathbf{i}+2\mathbf{j}$, $\vec{AC}=-3\mathbf{j}+2\mathbf{k}$, and $\vec{AD}=3\mathbf{i}-4\mathbf{j}+5\mathbf{k}$,

$$\begin{aligned} \left(\vec{AB}\times\vec{AC}\right)\bullet\vec{AD} &= \vec{PQ}\times\vec{PR}=\langle 1,2,1\rangle\times\langle -2,3,-3\rangle \\ &= \begin{vmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 3 & -4 & 5 \end{vmatrix} \\ &= \begin{vmatrix} -3 & 2 \\ -4 & 5 \end{vmatrix}-2\begin{vmatrix} 0 & 2 \\ 3 & 5 \end{vmatrix}+0\begin{vmatrix} 0 & 2 \\ 3 & 5 \end{vmatrix} \\ &= -15+8+12 \\ &= \boxed{5} \end{aligned}$$

Volume is now equal to $|\left(\vec{AB}\times\vec{AC}\right)\bullet\vec{AD}|=5$.

p.687, pr.17(a)



(c) 10 Points Use Integral Test to determine if the series $\sum_{n=2}^{\infty}\frac{\ln n}{n}$ converges or diverges. Be sure to check the conditions of the Integral Test are satisfied.

Solution: The function $f(x)=\frac{\ln x}{x}$ is positive, continuous, and decreasing for $x\geq 3$ so integral test applies.

$$\int_2^{\infty}\frac{\ln x}{x}\,\mathrm{d}x=\lim_{b\to\infty}\int_2^{b}\frac{\ln x}{x}\,\mathrm{d}x=\lim_{b\to\infty}\int_{\ln 2}^{\ln b}u\,\mathrm{d}u=\lim_{b\to\infty}\left[\frac{1}{2}u^2\right]_{\ln 2}^{\ln b}=\lim_{b\to\infty}\left[\frac{1}{2}(\ln b)^2-\frac{1}{2}(\ln 2)^2\right]=\infty-\frac{1}{2}(\ln 2)^2=\infty$$

Therefore, the integral $\int_2^{\infty}\frac{\ln x}{x}\,\mathrm{d}x$ diverges. This shows by the Integral Test, the series $\sum_{n=2}^{\infty}\frac{\ln n}{n}$ diverges.

p.600, pr.19

3. (a) [10 Points] Find the *Maclaurin series* for the function $f(x) = \dfrac{1}{1+x}$. Write its radius of convergence.

Solution: Using geometric series formula, we have

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots \quad \text{for } -1 < x < 1.$$

The radius of convergence is clearly $\boxed{1}$

p.632, pr.13

(b) [10 Points] Evaluate the integral $\int x(\ln x)\,\mathrm{d}x$.

Solution: We integrate by parts. Let $u = \ln x$ and $\mathrm{d}v = x\,\mathrm{d}x$. Then $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$ and choose $v = \frac{1}{2}x^2$. Hence

$$\begin{aligned}\int x \ln x\,\mathrm{d}x &= \int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u \\ &= (\ln x)(\tfrac{1}{2}x^2) - \int \left(\frac{1}{2}x^2\right)\left(\frac{1}{x}\right)\mathrm{d}x \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,\mathrm{d}x = \boxed{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}\end{aligned}$$

p.468, pr.33

(c) [13 Points] Evaluate the integral $\int \dfrac{(1-x^2)^{1/2}}{x^4}\,\mathrm{d}x$. (*Hint*: Use trigonometric substitution)

Solution: For $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, let $x = \sin\theta$ and so $\mathrm{d}x = \mathrm{d}\cos\theta\,\mathrm{d}\theta$. Then $(1-x^2)^{1/2} = (1-(\sin\theta)^2)^{1/2} = (1-\sin^2\theta)^{1/2} = (\cos^2\theta)^{1/2} = \cos\theta$. Then

$$\begin{aligned}\int \frac{(1-x^2)^{1/2}}{x^4}\,\mathrm{d}x &= \int \frac{\cos\theta}{\sin^4\theta}\cos\theta\,\mathrm{d}\theta = \int \frac{\cos^2\theta}{\sin^2\theta}\frac{1}{\sin^2\theta}\,\mathrm{d}\theta = \int \cot^2\theta \csc^2\theta\,\mathrm{d}\theta = -\int y^2\,\mathrm{d}y = -\frac{1}{3}y^3 + c \\ &= -\frac{1}{3}\cot^3\theta + c = \boxed{-\frac{1}{3}\left(\frac{(1-x^2)^{1/2}}{x}\right)^3 + c}\end{aligned}$$

p.481, pr.28