Math 114 Summer 2016 **Final Exam** August 27, 2016

Your Name / Ad - Soyad (**90 min.**) Signature / İmza

Student ID # / Öğrenci No (**mavi tükenmez!**)

Problem	1	2	3	4	Total
Points:	40	25	26	24	115
Score:					

You have **90 minutes**. **(Cell phones off and away!)**. No books, notes or calculators are permitted. **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

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1. (a) (10 Points) Evaluate the integral $\int x^3 \mathrm{e}^{(x^2)}\,\mathrm{d}x$.

Solution: Let $y = x^2$. Then $dy = 2x\ \mathrm{d}x$. Therefore

$$\begin{aligned}\int x^3 \mathrm{e}^{(x^2)}\,\mathrm{d}x &= \frac{1}{2}\int x^2 \mathrm{e}^{(x^2)} 2x\,\mathrm{d}x \\ &= \frac{1}{2}\int y\mathrm{e}^y\ \mathrm{d}y\end{aligned}$$

We now integrate by parts. For let $u = y$ and so $dv = \mathrm{e}^y\ \mathrm{d}y$/ Then $du = dy$ and choose $v = \mathrm{e}^y$. Hence

$$\begin{aligned}\int y\mathrm{e}^y\ \mathrm{d}y = \int u\ \mathrm{d}v &= uv - \int v\ \mathrm{d}u \\ &= y\mathrm{e}^y - \int \mathrm{e}^y\ \mathrm{d}y \\ &= y\mathrm{e}^y - \mathrm{e}^y + C_1\end{aligned}$$

Finally, we have

$$\int x^3 \mathrm{e}^{(x^2)}\,\mathrm{d}x = \frac{1}{2}\left(x^2\mathrm{e}^{x^2} - \mathrm{e}^{x^2} + C_1\right)$$

p.491, pr.86

(b) (10 Points) Find the parametric equations of the line normal to the surface $x^2z + xz^2 + y^2 = yz + 5x + 5$ at the point $P_0(1,2,3)$.

Solution: Let $F(x,y,z) = x^2z + xz^2 + y^2 - yz - 5x - 5 = 0$. Then

$$\begin{aligned}F_x &= 2xz + z^2 - 5 \\ F_y &= 2y - z \\ F_z &= x^2 + 2xz - y.\end{aligned}$$

Then

$$\begin{aligned}F_x(1,2,3) &= 6 + 9 - 5 = 10 \\ F_y(1,2,3) &= 4 - 3 = 1 \\ F_z(1,2,3) &= 1 + 6 - 2 = 5.\end{aligned}$$

The parametric equations for required normal line are

$$\boxed{x = 1 + 8t, \qquad y = 2 + t, \qquad z = 3 + 5t}$$

p.491, pr.86

(c) (10 Points) Evaluate the integral $\int \frac{\ln x}{x + x\ln x}\,\mathrm{d}x$.

Solution: Let $y = \ln x$ and so $\mathrm{d}y = \frac{1}{y}\ \mathrm{d}y$. Then

$$\begin{aligned}\int \frac{\ln x}{x + x\ln x}\,\mathrm{d}x &= \int \frac{\ln x}{1 + \ln x}\frac{1}{x}\,\mathrm{d}x \\ &= \int \frac{y}{1+y}\,\mathrm{d}y \\ &= \int \frac{y+1-1}{1+y}\,\mathrm{d}y \\ &= \int \left(1 - \frac{1}{1+y}\right)\mathrm{d}y \\ &= \boxed{y - \ln|1+y| + C = \ln x - \ln|1 + \ln x| + C}\end{aligned}$$

p.491, pr.107

(d) (10 Points) Use logarithmic differentiation to find the derivative of

$$y = \sqrt[3]{\frac{x(x+1)(x+2)}{(x^2+1)(2x+3)}}$$

with respect to x.

Solution:

$$\begin{aligned}\ln y &= \frac{1}{3}\left[\ln x + \ln(x+1) + \ln(x+2) - \ln(x^2+1) - \ln(x^2+1)\right. \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{3}\left[\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2} - \frac{2x}{x^2+1} - \frac{2}{2x+3}\right] \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y}{3}\left[\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2} - \frac{2x}{x^2+1} - \frac{2}{2x+3}\right] \\ &= \frac{1}{3}\sqrt[3]{\frac{x(x+1)(x+2)}{(x^2+1)(2x+3)}}\left[\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2} - \frac{2x}{x^2+1} - \right.\end{aligned}$$

p.94, pr.34

2. Given the three points $P(1,0,1)$, $Q(2,0,0)$, and $R(-1,2,2)$.

(a) (6 Points) Find the area of the triangle having P, Q and R as vertices.

Solution: First form the vectors $\vec{PQ} = (2-1)\mathbf{i} + (0-0)\mathbf{j} + (0-1)\mathbf{k} = \mathbf{i} - \mathbf{k}$ and $\vec{PR} = (-1-1)\mathbf{i} + (2-0)\mathbf{j} + (2-1)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$$A = \frac{1}{2}\|\vec{PQ} \times \vec{PR}\|$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ -2 & 2 & 1 \end{vmatrix}$$

$$= 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

$$\|\vec{PQ} \times \vec{PR}\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$$

$$A = \frac{3}{2}$$

p.551, pr.32

(b) (6 Points) Find the angle (in degrees) at the vertex P of the triangle having P, Q and R as vertices.

Solution: Let θ denote the angle we want to find. Then

$$\begin{aligned} \cos\theta &= \frac{\vec{PQ} \cdot \vec{PR}}{\|\vec{PQ}\|\|\vec{PR}\|} \\ &= \frac{(\mathbf{i} - \mathbf{k}) \cdot (-2\mathbf{i} + 2\mathbf{j} + \mathbf{k})}{\sqrt{2}\sqrt{9}} \\ &= \frac{1}{3\sqrt{2}}((1)(-2) + (0)(2) + (-1)(1)) \\ &= \frac{-3}{3\sqrt{2}} = \boxed{-\frac{1}{\sqrt{2}}} \end{aligned}$$

Hence $\boxed{\theta = 135}$ degrees.

p.551, pr.32

(c) (6 Points) Find the equation of the plane containing the points P, Q and R.

Solution: We know from part (a) that the vector

$$\mathbf{n} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

is normal to the plane. Therefore

$$\mathbf{n} \cdot \vec{P_0P} =$$

gives

$$(2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \cdot ((x-1)\mathbf{i} + (y-0)\mathbf{j} + (z-1)\mathbf{k}) = 0$$

Hence $2(x-1) + y + 2(z-1) = 0 \Rightarrow 2x + y + 2z = 4$

p.551, pr.32

(d) (7 Points) Find the value(s) of c if the function

$$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ c, & (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$.

Solution: We employ the polar coordinates: $x = r\cos\theta$ and $y = r\sin\theta$. Then $x^2 + y^2 = r^2$ and $r \to 0$ as $(x,y) \to (0,0)$. Hence we have

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{r\to 0} \frac{3r^3\cos^2\theta\sin\theta}{r^2} \\ &= \lim_{r\to 0}(3r\cos^2\theta\sin\theta) \\ &= 0 \end{aligned}$$

So $f(x,y)$ is continuous at $(0,0)$ iff $\boxed{c=0}$.

p.82, pr.35

3. (a) (10 Points) Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$ when $u = \ln 2$, $v = 1$ if $z = 5\tan^{-1} x$ and $x = e^u + \ln v$.

Solution: We apply the two chain rule formulas.

$$\frac{\partial z}{\partial u} = \frac{dz}{dx}\frac{\partial x}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{dz}{dx}\frac{\partial x}{\partial v}.$$

Differentiating gives

$$\frac{dz}{dx} = \frac{5}{1+x^2}, \qquad \frac{\partial x}{\partial u} = \mathrm{e}^u, \qquad \frac{\partial x}{\partial v} = \frac{1}{v}.$$

Moreover, when $u = \ln 2$ and $v = 1$, we have $x = e^{\ln 2} + \ln 1 = 2 + 0 = 2$. And derivatives at these points have values:

$$\left.\frac{dz}{dx}\right|_{x=2} = \frac{5}{1+2^2} = 1, \qquad \left.\frac{\partial x}{\partial u}\right|_{u=\ln 2} = \mathrm{e}^{\ln 2} = 2, \qquad \left.\frac{\partial x}{\partial v}\right|_{v=1} = \frac{1}{1} = 1.$$

Therefore

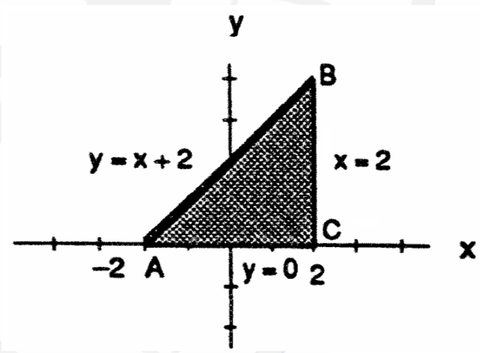
$$\boxed{\left.\frac{\partial z}{\partial u}\right|_{\substack{u=\ln 2\\ v=1}} = (1)(2) = 2}, \qquad \boxed{\left.\frac{\partial z}{\partial v}\right|_{\substack{u=\ln 2\\ v=1}} = (1)(1) = 1}.$$

p.72, pr.8

(b) (16 Points) Find the absolute maximum and minimum values of $f(x,y) = x^2 - y^2 - 2x + 4y$ on the given region R.

Solution:
• *Interior Points of this triangular region R*: $f_x(x,y) = 2x - 2 = 0 \Rightarrow x = 1$ and $f_y(x,y) = -2y + 4 = 0 \Rightarrow y = 2 \Rightarrow (1,2)$ is an interior critical point of R with $f(1,2) = 3$.
• *On AB*, we have $f(x, x+2) = -2x+4$ for $-2 \le x \le 2$. So $f'(x) = -2 = 0 \Rightarrow$ no critical points in the interior of AB. Endpoints of AB: $f(-2,0) = 8$ and $f(2,4) = 0$.
• *On BC*, we have $f(x,y) = f(2,y) = -y^2 + 4y$ for $0 \le y \le 4$. So $f'(2,y) = -2y + 4 = 0 \Rightarrow y = 2$ and $x = 2 \Rightarrow (2,2)$ is an interior critical point of BC with $f(2,2) = 4$. Endpoints of BC: $f(2,0) = 0$ and $f(2,4) = 0$.
• *On AC*, we have $f(x,y) = f(x,0) = x^2 - 2x$ for $-2 \le x \le 2$. So $f'(x,0) = 2x - 2 = 0 \Rightarrow x = 1$ and $y = 0 \Rightarrow (1,0)$ is an interior critical point of AC with $f(1,0) = -1$. Endpoints of AC: $f(-2,0) = 8$ and $f(2,0) = 0$.
• Therefore the absolute maximum is 8 at $(-2,0)$ and the absolute minimum is -1 at $(1,0)$,

p.317, pr.33



4. (a) (10 Points) Use the fact that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ to find the *first three nonzero terms* of the Maclaurin series for $\int \frac{e^x - 1}{x}\, dx$.

Solution: Using

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + + \frac{x^5}{5!} + \cdots, \quad -\infty < x < \infty$$

we have

$$\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots, \quad -\infty < x < \infty$$

This is the Mclaurin series for $\frac{e^x - 1}{x}$. We can now do term-by-term integration.

$$\int \left(\frac{e^x - 1}{x} \right) dx = \int \left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots \right) dx, \quad -\infty < x < \infty.$$

So we get

$$\boxed{\int \left(\frac{e^x - 1}{x} \right) dx = C + x + \frac{x^2}{2 \times 2!} + \frac{x^3}{3 \times 3!} + \frac{x^4}{4 \times 4!} + \frac{x^5}{5 \times 5!} + \cdots, \quad -\infty < x < \infty.}$$

as required. Note that the radius of convergence is $R = \infty$.

p.95, pr.68

(b) (14 Points) Find the *radius and interval of convergence* for the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{n2^n}$.

Solution: Let $u_n = \frac{(-1)^{n+1}(x+2)^n}{n2^n}$. Then $u_{n+1} = \frac{(-1)^{n+2}(x+2)^{n+1}}{(n+1)2^{n+1}}$ and so

$$\frac{u_{n+1}}{u_n} = \frac{\cancel{(-1)^{n+1}}(-1)\cancel{(x+2)^n}(x+2)}{(n+1)\cancel{2^n}2} \cdot \frac{n\cancel{2^n}}{\cancel{(-1)^{n+1}}\cancel{(x+2)^n}} = -\frac{1}{2}\frac{n}{n+1}(x+2).$$

Therefore, the power series converges absolutely if

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n\to\infty} \left| -\frac{1}{2}\frac{n}{n+1}(x+2) \right| < 1 \Rightarrow \frac{|x+2|}{2} \lim_{n\to\infty} \frac{n}{n+1} < 1 \Rightarrow \frac{|x+2|}{2} \underbrace{\lim_{n\to\infty} \frac{1}{1+1/n}}_{1} < 1 \Rightarrow |x+2| < 2,$$

that is, if $-2 < x+2 < 2$, or if, $-4 < x < 0$. Now the endpoints are -4 and 0. We shall test the series for convergence at these points. When $x = -4$, we have $\sum_{n=1}^{\infty} \frac{-1}{n}$, a divergent series; when $x = 0$, we have $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, the alternating harmonic series which converges conditionally. Therefore the radius of convergence is $\boxed{R=2}$ and the interval of convergence is $\boxed{-4 < x \leq 0}$.

p.583, pr.39