Math 114 Summer 2018 **Final Exam** August 10, 2018

Your Name / Ad - Soyad (**90 min.**) Signature / İmza

Student ID # / Öğrenci No (**mavi tükenmez!**)

Problem	1	2	3	4	Total
Points:	24	32	30	24	110
Score:					

Time limit is **90 minutes**. If you need more room on your exam paper, you may use the empty spaces on the paper. You have to show your. Answers without reasonable-even if your results are true- work will either get zero or very little credit.

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1. (a) (12 Points) Evaluate the integral $\int_0^{\ln 4} \frac{e^t\, dt}{\sqrt{e^{2t}+9}}$.

Solution: Let $e^t = 3\tan\theta$, $t = \ln(3\tan\theta)$, $\tan^{-1}(\frac{1}{3}) \le \theta \le \tan^{-1}(\frac{4}{3})$, $dt = \frac{\sec^2\theta}{\tan\theta}d\theta$, $\sqrt{e^{2t}+9} = \sqrt{9\tan^2\theta+9} = 3\sec\theta$;

$$\int_0^{\ln 4} \frac{e^t\, dt}{\sqrt{e^{2t}+9}} = \int_{\tan^{-1}(\frac{1}{3})}^{\tan^{-1}(\frac{4}{3})} \frac{\tan\theta\sec^2\theta\, d\theta}{\tan\theta\cdot 3\sec\theta} = \int_{\tan^{-1}(\frac{1}{3})}^{\tan^{-1}(\frac{4}{3})} \sec\theta\, d\theta = [\ln|\sec\theta+\tan\theta|]_{\tan^{-1}(\frac{1}{3})}^{\tan^{-1}(\frac{4}{3})}$$

$$= \ln\left(\frac{5}{3}+\frac{4}{3}\right) - \ln\left(\frac{\sqrt{10}}{3}+\frac{1}{3}\right) = \boxed{\ln(9) - \ln\left(1+\sqrt{10}\right)}$$

p.72, pr.8

(b) (12 Points) For <u>what values of x</u> does the series $1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-3)^n + \cdots$ converge? What is its <u>sum</u>? If you <u>differentiate</u> this series term-by-term, what series do you get? For what values of x will the new series converge?

Solution: $\lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{2^{n+1}}\cdot\frac{2^n}{(x-3)^n}\right| < 1 \Rightarrow |x-3| < 2 \Rightarrow 1 < x < 5$; when $x = 1$ we have $\sum_{n=1}^{\infty}(1)^n$ which diverges; when $x = 5$ we have $\sum_{n=1}^{\infty}(-1)^n$ which also diverges; the interval of convergence is $1 < x < 5$; the sum of this convergent series is $\frac{1}{1+\frac{x-3}{2}} = \frac{2}{x-1}$. If $f(x) = 1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-3)^n + \cdots = \frac{2}{x-1}$, then $f'(x) = -\frac{1}{2} + \frac{1}{2}(x-3) + \cdots + \left(-\frac{1}{2}\right)^n n(x-3)^{n-1} + \cdots =$ is convergent when $1 < x < 5$, and diverges when $x = 1$ or $x = 5$. The sum for $f'(x)$ is $\frac{-2}{(x-1)^2}$, the derivative of $\frac{2}{x-1}$.

p.72, pr.8

2. (a) (10 Points) Write the <u>parametric equations for the line</u> that passes through $(0,-7,0)$ and perpendicular to the plane $x+2y+2z=13$.

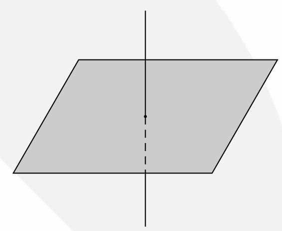
Solution: The direction is $\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ and $P(0,7,0) \Rightarrow \mathscr{L}: \boxed{\begin{cases} x=t \\ y=-7+2t \\ z=2t \end{cases}}$

p.72, pr.8

(b) (10 Points) Find the <u>point of intersection</u> of the line $\mathscr{L}: \begin{cases} x=1-t, \\ y=3t, \\ x=1+t \end{cases}$ and the plane $\mathscr{M}: 6x+3y-4z=-12$.

Solution: $6x+3y-4z=-12 \Rightarrow 6(2)+3(3+2t)-4(-2-2t)=-12 \Rightarrow 14t+29=-12 \Rightarrow t=-\frac{41}{14}$.
So $x=2, \quad y=3-\frac{41}{7}, \quad z=-2+\frac{41}{7} \Rightarrow \mathscr{L}\cap\mathscr{M} = \{(2,-\frac{20}{7},\frac{27}{7})\}$ is the point of intersection.

p.72, pr.8

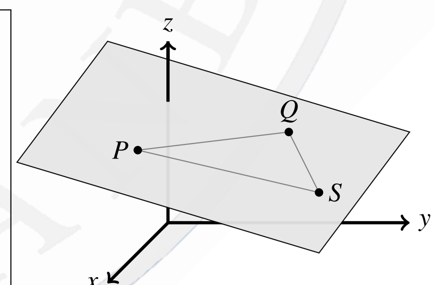


(c) (12 Points) Write an <u>equation for the plane</u> passing through the points $P(1,-1,2)$, $Q(2,1,3)$ and $S(-1,2,-1)$.

Solution: First $\vec{PQ}=(2-1)\mathbf{i}+(1+1)\mathbf{j}+(3-2)\mathbf{k}=\mathbf{i}+2\mathbf{j}+\mathbf{k}$ and $\vec{PS}=(-1-1)\mathbf{i}+(2+1)\mathbf{j}+(-1-2)\mathbf{k}=-2\mathbf{i}+3\mathbf{j}-3\mathbf{k}$. Then

$$\vec{PQ}\times\vec{PS}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ -2 & 3 & -3 \end{vmatrix}$$
$$=\begin{vmatrix} 2 & 1 \\ 3 & -3 \end{vmatrix}\mathbf{i}-\begin{vmatrix} 2 & 1 \\ -2 & -3 \end{vmatrix}\mathbf{j}+\begin{vmatrix} 2 & 2 \\ -2 & -3 \end{vmatrix}\mathbf{k}$$
$$=-9\mathbf{i}+4\mathbf{j}+10\mathbf{k}$$
$$\Rightarrow \text{Plane Eq. } -9(x-1)+4(y+1)+10(z-2)=0 \Rightarrow \boxed{-9x+4y+10z=7}$$

p.687, pr.17(a)



3. (a) (10 Points) Show that the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4-y^2}{x^4+y^2}$ does not exist.

Solution: We consider the parabolas $y = kx^2$. Then

$$\lim_{(x,kx^2)\to(0,0)} \frac{x^4-(kx^2)^2}{x^4+(kx^2)^2} = \lim_{x\to 0} \frac{x^4-k^2x^4}{x^4+k^2x^4} = \lim_{x\to 0} \frac{\cancel{x^4}(1-k^2)}{\cancel{x^4}(1+k^2)} = \frac{1-k^2}{1+k^2}$$

So we have different limits for different values of k. Hence the original limit does not exist.

p.83, pr.52

(b) (10 Points) Suppose $z = \ln q$ and $q = \sqrt{v+3}\tan^{-1} u$. Find the values of $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$ when $u = 1$ and $v = -2$.

Solution:

$$\frac{\partial z}{\partial v} = \frac{dz}{dq}\frac{\partial q}{\partial v} = \frac{1}{q}\frac{\tan^{-1}u}{2\sqrt{v+3}} = \frac{1}{\sqrt{v+3}\cancel{\tan^{-1}u}}\frac{\cancel{\tan^{-1}u}}{2\sqrt{v+3}} = \frac{1}{2(v+3)}$$
$$\frac{\partial z}{\partial u} = \frac{dz}{dq}\frac{\partial q}{\partial u} = \frac{1}{q}\frac{\sqrt{v+3}}{1+u^2} = \frac{1}{\cancel{\sqrt{v+3}}\tan^{-1}u}\frac{\cancel{\sqrt{v+3}}}{1+u^2} = \frac{1}{(1+u^2)\tan^{-1}u}$$

Now we evaluate the derivatives at $(u,v) = (-1,-2)$:

$$\left.\frac{\partial z}{\partial v}\right|_{(-1,-2)} = \left.\frac{1}{2(v+3)}\right|_{(-1,-2)} = \frac{1}{2(-2+3)} = \boxed{\frac{1}{2}}$$
$$\left.\frac{\partial z}{\partial u}\right|_{(-1,-2)} = \left.\frac{1}{(1+u^2)\tan^{-1}u}\right|_{(-1,-2)} = \frac{1}{(1+(-1)^2)\tan^{-1}(1)} = \frac{1}{2\frac{\pi}{4}} = \boxed{\frac{2}{\pi}}$$

p.72, pr.8

(c) (10 Points) Write the equation for the plane tangent to the surface $x^2+2xy-y^2+z^2 = 7$ at the point $P_0(1,-1,3)$.

Solution: Let $F(x,y,z) = x^2+2xy-y^2+z^2-7 = 0$. Then

$$\nabla F(x,y,z) = F_x(x,y,z)\mathbf{i} + F_y(x,y,z)\mathbf{j} + F_x(x,y,z)\mathbf{k} = (2x+2y)\mathbf{i} + (2x-2y)\mathbf{j} + (-2z)\mathbf{k}$$
$$\nabla F(1,-1,3) = (2-2)\mathbf{i} + (2+2)\mathbf{j} - 6\mathbf{k}$$
$$= 4\mathbf{j} - 6\mathbf{k}$$

Hence the equation for the tangent plane is

$$4(y+1) - 6(z-3) = 0 \Rightarrow 4y - 6z = -22 \Rightarrow \boxed{2y - 3z = -11}$$

p.83, pr.52

4. (a) (12 Points) Let $f(x,y) = xy^2 - 2x^2y + 4x^3 - 9x$. Find all the critical points of $f(x,y)$ and then determine if it gives a local max, a local min or a saddle point.

Solution: $f_x(x,y) = y^2 - 4xy + 12x^2 - 9 = 0$ and $f_y(x,y) = 2xy - 2x^2 = 0 \Rightarrow 2x(y-x) = 0 \Rightarrow x = 0$ or $y = x$. when $x = 0$, we have $y^2 - 9 = 0 \Rightarrow y = \pm 3 \Rightarrow (0,3), (0,-3)$.

Next $y = x$ gives $x^2 - 4x^2 + 12x^2 - 9 = 0 \Rightarrow 9x^2 - 9 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$.

Thus CP's are $(x,y) = (0,3)$, $(0,-3)$, $(1,1)$, $(-1,-1)$.

Next $f_{xx} = -4y + 24x$, $f_{yy} = -4x$, $f_{xy} = 2y - 4x$.

$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = (-4y+24x)(2y-4x) - (-4x)^2$.

(x,y)	$D(x,y) = (-4y+24x)(2y-4x) - (-4x)^2$
$(0,3)$	$D(0,3) = 0 - 36 < 0 \to$ SADDLE POINT $f(0,3) = 0$
$(0,-3)$	$D(0,-3) = 0 - 36 < 0 \to$ SADDLE POINT $f(0,-3) = 0$
$(1,1)$	$D(1,1) = 40 - 4 = 36 > 0$ and $f_{xx}(1,1) = 20 > 0 \to$ LOCAL MIN $f(1,1) = -6$
$(-1,-1)$	$D(-1,-1) = 40 - 4 = 36 > 0$ and $f_{xx}(-1,-1) = -20 <$ LOCAL MAX $f(-1,-1) = 7$

p.82, pr.35

(b) (12 Points) Use the method of Lagrange Multipliers to find the point on the surface $z^2 = xy + 4$ that is closest to the origin.

Solution: Let $f(x,y,z) = x^2 + y^2 + z^2$ be the square of the distance to the origin and we wish to minimize it subject to the constraint $g(x,y,z) = z^2 = xy - 4 = 0$. Then

$\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = -y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k}$ so that

$\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(-y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k}) \Rightarrow 2x = -y\lambda 2x = -y\lambda,\ 2y = -x\lambda,\ 2z = 2z\lambda$
$\Rightarrow 2z(1-\lambda) = 0 \Rightarrow z = 0$ or $\lambda = 0$

CASE 1: $\lambda = 1 \Rightarrow 2x = -y$ and $2y = -x \Rightarrow y = 0$ and $x = 0 \Rightarrow z^2 - 4 = 0 \Rightarrow z = \pm 2$ and $x = y = 0$.

CASE 2: $z = 0 \Rightarrow -xy - 4 = 0 \Rightarrow y = -\frac{4}{x}$.
Then $2x = \frac{4}{x}\lambda \Rightarrow \lambda = \frac{x^2}{2}$, and $-\frac{8}{x} = -x\lambda \Rightarrow -\frac{8}{x} = -x(\frac{x^2}{2}) \Rightarrow x^4 = 16 \Rightarrow x = \pm 2$.

Thus, $x = 2$ and $y = 2$, or $x = -2$ and $y = +2$. Therefore, we get four points: $(2,-2,0)$, $(-2,2,0)$, $(0,0,2)$, and $(0,0,-2)$. Now we evaluate f at these four points.

(x,y,z)	$f(x,y,z) = x^2 + y^2 + z^2$
$(2,-2,0)$	$f(2,-2,0) = 8 \to$ distance to origin $= \sqrt{8} = 2\sqrt{2}$
$(-2,2,0)$	$f(-2,2,0) = 8 \to$ distance to origin $= \sqrt{8} = 2\sqrt{2}$
$(0,0,2)$	$f(0,0,2) = 4 \to$ distance to origin $= \sqrt{4} = 2$
$(0,0,-2)$	$f(0,0,-2) = 4$ distance to origin $= \sqrt{4} = 2$

Hence $(0,0,2)$, and $(0,0,-2)$ are the closest points to the origin since they are 2 units away the others are $2\sqrt{2}$ units away.

p.688, pr.48