Math 114 Summer 2019 **Final Exam** **August 1, 2019**

Your Name / Ad - Soyad ______ (**70 min.**)

Signature / İmza ______

Student ID # / Öğrenci No ______ (**mavi tükenmez!**)

Problem	1	2	3	4	Total
Points:	26	22	25	27	100
Score:					

Time limit is **70 minutes**. If you need more room on your exam paper, you may use the empty spaces on the paper. You have to show your. Answers without reasonable-even if your results are true- work will either get zero or very little credit.

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1. (a) (10 Points) Evaluate the integral $\int_4^8 \frac{y\,dy}{y^2-2y-3}$.

Solution: By using partial fractions, we have

$$\frac{y}{y^2-2y-3} = \frac{A}{y-3} + \frac{B}{y+1} \Rightarrow y = A(y+1)+B(y-3); y=-1 \Rightarrow B = \frac{-1}{-4} = \frac{1}{4}; y=3 \Rightarrow A = \frac{3}{4};$$

$$\int_4^8 \frac{y\,dy}{y^2-2y-3} = \frac{3}{4}\int_4^8 \frac{dy}{y-3} + \frac{1}{4}\int_4^8 \frac{dy}{y+1} = \left[\frac{3}{4}\ln|y-3| + \frac{1}{4}\ln|y+1|\right]_4^8$$

$$= \left(\frac{3}{4}\ln 5 + \frac{1}{4}\ln 9\right) - \left(\frac{3}{4}\ln 1 + \frac{1}{4}\ln 5\right) = \frac{1}{2}\ln 5 + \frac{1}{2}\ln 3 = \boxed{\frac{\ln(15)}{2}}$$

p.94, pr.34

(b) (8 Points) Assuming that the equation $x^2+xy+y^2-7=0$ defines y as a differentiable function of x, find the value of $\frac{dy}{dx}$ at the point $(1,2)$.

Solution: Let $F(x,y) = x^2+xy+y^2-7=0$. Then

$$F_x(x,y) = 2x+y \text{ and } F_y(x,y) = x+2y$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x+y}{x+2y}$$

$$\frac{dy}{dx}(1.2) = \boxed{-\frac{4}{5}}$$

p.94, pr.34

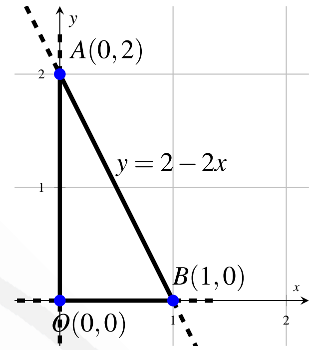
(c) (8 Points) Find $\partial w/\partial r$ when $r=1$ and $s=-1$, if $w=(x+y+z)^2$, $x=r-s$, $y=\cos(r+s)$, $z=\sin(r+s)$.

Solution: By the Chain Rule formula,

$$\begin{aligned}
\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\
&= 2(x+y+z)(1) + 2(x+y+z)(-\sin(r+s)) + 2(x+y+z)(\cos(r+s)) \\
&= 2(x+y+z)[1-\sin(r+s)+\cos(r+s)] \\
&= 2(r-s+\cos(r+s)+\sin(r+s))[1-\sin(r+s)+\cos(r+s)] \\
&= \left.\frac{\partial w}{\partial r}\right|_{r=1,s=-1} = (2)(3(2) = \boxed{12}
\end{aligned}$$

p.94, pr.34

2. (a) (11 Points) Find the absolute maxima and minima of $f(x,y) = x^2 + y^2$ on the closed triangular plate bounded by the lines $x = 0$, $y = 0$ and $y + 2x = 2$.



Solution: Let the vertices be $A(0,2)$, $B(1,0)$, $O(0,0)$.

Along OA, $f(x,y) = f(0,y) = y^2$ on $0 \le y \le 2$;
$f'(0,y) = 2y = 0 \Rightarrow y = 0$ and $x = 0$;
$f(0,0) = 0$ and $f(0,2) = 4$.

Along OB, $f(x,y) = f(x,0) = x^2$ on $0 \le x \le 1$;
$f'(x,0) = 2x \Rightarrow x = 0$ and $y = 0$
$f(0,0) = 0$ and $f(1,0) = 1$.

Along AB, $f(x,y) = f(x,-2x+2) = 5x^2 - 8x + 4$ on $0 \le x \le 1$;
$f'(x,-2x+2) = 16x - 8 = 0 \Rightarrow x = \frac{4}{5}$ and $y = \frac{2}{5}$; $f(\frac{4}{5},\frac{2}{5}) = \frac{4}{5}$;
endpoint values have been found above.

For interior points, $f_x(x,y) = 2x = 0$ and $f_y(x,y) = 2y = 0 \Rightarrow x = 0$ and $y = 0$ but $(0,0)$ is not an interior point of the region. Therefore the absolute maximum is 4 at $(0,2)$ and the absolute minimum is 0 at $(0,0)$.

p.192, pr.87

(b) (11 Points) Use Lagrange Multipliers to find the points on the surface $z^2 - xy = 4$ closest to the origin.

Solution: Let $f(x,y,z) = x^2 + y^2 + z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = -y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(-y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k}) \Rightarrow 2x = -y\lambda$, $2y = -x\lambda$, and $2z = 2z\lambda \Rightarrow \lambda = 1$ or $z = 0$.

CASE 1: $\lambda = 1 \Rightarrow 2x = -y$ and $2y = -x \Rightarrow y = 0$ and $x = 0 \Rightarrow z^2 - 4 = 0 \Rightarrow z = \pm 2$ and $x = y = 0$.

CASE 2: $z = 0 \Rightarrow -xy - 4 = 0 \Rightarrow y = -\frac{4}{x}$. Then $2x = \frac{4}{x}\lambda \Rightarrow \lambda = \frac{x^2}{2}$, and $-\frac{8}{x} = -\frac{x}{\lambda} \Rightarrow -\frac{8}{x} = -x\left(\frac{x^2}{2}\right) \Rightarrow x^4 = 16 \Rightarrow x = \pm 2$. Thus $x = 2$ and $y = -2$ or $x = -2$ and $y = 2$.
Therefore we get four points: $(2,-2,0)$, $(-2,2,0)$, $(0,0,2)$. and $(0,0,-2)$. But *the points* $(0,0,2)$ *and* $(0,0,-2)$ *are closest to the origin* since they are 2 units away and the others are $2\sqrt{2}$ units away.

p.82, pr.35

3. (a) (10 Points) Use vectors to find the area of the triangle with vertices $A(-5,3)$, $B(1,-2)$, $C(6,-2)$.

Solution: We first form the vectors

$$\vec{AB} = 6\mathbf{i} - 5\mathbf{j} \quad \vec{AC} = 11\mathbf{i} - 5\mathbf{j}$$

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -5 & 0 \\ 11 & -5 & 0 \end{vmatrix} = \begin{vmatrix} -5 & 0 \\ -5 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 6 & 0 \\ 11 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 6 & -5 \\ 11 & -5 \end{vmatrix}\mathbf{k}$$

$$= 25\mathbf{k}$$

Hence area $= \frac{1}{2}|\vec{AB}\times\vec{AC}| = \frac{1}{2}|25\mathbf{k} = \boxed{\frac{25}{2}}$

p.95, pr.68

(b) (7 Points) Write the equation for the plane through $(1,-1-3)$ and parallel to the plane $3x+y+z=7$.

Solution: The plane has equation

$$3(x-1)+(1)(y+1)+(1)(z+3)=0 \Rightarrow \boxed{3x+y+z=5}$$

p.112, pr.26

(c) (8 Points) Find point of intersection of the line $x=1-t$, $y=3t$, $z=1+t$ and the plane $2x-y+3z=6$.

Solution: Substitute the equations for the line into the equation for the plane

$$2x-y+3z=6 \Rightarrow 2(1-t)-(3t)+3(1+t)=6 \Rightarrow -2t+5=6 \Rightarrow t=-\frac{1}{2} \Rightarrow x=-\frac{3}{2}, y=\frac{3}{2}, z=\frac{1}{2}$$

Therefore $\boxed{\left(-\frac{3}{2},\frac{3}{2},\frac{1}{2}\right)}$ is the point of intersection.

p.112, pr.26

4. (a) (10 Points) Find equations for the (a) tangent plane and (b) normal line at the point $P_0(1,-1,3)$ on the surface $x^2+2xy-y^2+z^2=7$.

Solution: First find the gradient.

$$\text{(a) } \nabla f(x,y,z) \text{ (a) } = (2x+2y)\mathbf{i}+(2x-2y)\mathbf{j}+(2x)\mathbf{k} \Rightarrow \nabla f(1,-1,3) = 4\mathbf{j}+6\mathbf{k}$$

$$\Rightarrow \text{ Tangent Plane :} 4(y+1)+6(z-3)=0 \Rightarrow 4y+6z=14 \Rightarrow \boxed{2y+3z=7}$$

$$\text{(b) Normal Line : } \boxed{x=1, \quad y=-1+4t, \quad z=3+6t}$$

p.112, pr.26

(b) (8 Points) Use the Limit Comparison Test to investigate the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{n^3-n^2+3}$.

○ Converges. ○ Diverges. Test Used: ____________

Solution: Compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent p-series since $p=2>1$. Both series have positive terms for $n \geq 1$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n-2}{n^3-n^2+3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3-2n^2}{n^3-n^2+3} = \lim_{n\to\infty} \frac{3n^2-4n}{3n^2-2n} = \lim_{n\to\infty} \frac{6n-4}{6n-2} = \lim_{n\to\infty} \frac{6n}{6n} = 1 > 0.$$

Then by Limit Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{n^3-n^2+3}$ converges.

p.72, pr.8

(c) (9 Points) Find the first four nonzero terms in the Maclaurin series for the function $y=e^x \sin x$.

Solution: We know that

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

and

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

Hence

$$e^x \sin x = \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right) = \boxed{x+x^2+\frac{1}{3}x^3-\frac{1}{30}x^5-\cdots}$$

p.83, pr.52