Cep telefonunuzu gözetmene teslim ediniz / Deposit your cell phones to invigilator

Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- Calculators, cell phones off and away!.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 70 min.

Do not write in the table to the right.

Problem	Points	Score
1	32	
2	35	
3	33	
Total:	100	

1. Evaluate the following integrals.

(a) 10 Points $\int x^2 \cos x \, \mathrm{d}x =$

Solution: Let $u = x^2$ and so $dv = \cos x \, dx$. Then $du = 2x \, \mathrm{d}x$ and choose $v = \sin x \, \mathrm{d}x$. Therefore, we have

$$\int x^2 \cos x \, \mathrm{d}x = \int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u = (x^2)(\sin x) - \int (\sin x)(2x) \, \mathrm{d}x = x^2 \sin x - \int 2x \sin x \, \mathrm{d}x$$

Now to evaluate $\int 2x \sin x \, \mathrm{d}x$, let $u = 2x$ and $dv = \sin x \, \mathrm{d}x$. Then $du = 2 \, \mathrm{d}x$ and choose $v = -\cos x$. Hence we have

$$\int x^2 \cos x \, \mathrm{d}x = x^2 \sin x - \left[(2x)(-\cos x) - \int (-\cos x)(2) \, \mathrm{d}x\right] x^2 \sin x + 2x \cos x - 2 \int \cos x$$

$$= \boxed{x^2 \sin x + 2x \cos x - 2 \sin x + c}$$

p.652, pr.2

(b) 12 Points $\int \cos^3 x \sin^3 x \, \mathrm{d}x =$

Solution: Let $u = \sin x$ and so $du = \cos x * \mathrm{d}x$. Then

$$\int \cos^3 x \sin^3 x \, \mathrm{d}x = \int \sin^3 x \cos^2 x \cos x \, \mathrm{d}x = \int \sin^3 x (1 - \sin^2 x) \cos x \, \mathrm{d}x$$

$$= \int u^3 (1 - u^2) \, \mathrm{d}u = \int (u^3 - u^5) \, \mathrm{d}u = \frac{u^4}{4} - \frac{u^6}{6} + c$$

$$= \frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + c$$

p.573, pr.38

(c) 10 Points $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{4x^2 + 9} =$

○ Converges. ○ Diverges. Integral's value = ____________

Solution: Let $u = \tan x$ and so $du = \sec^2 x \, dx$. Then we have

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{4x^2 + 9} = 2 \int_0^{\infty} \frac{\mathrm{d}x}{4x^2 + 9} = \frac{1}{2} \int_0^{\infty} \frac{\mathrm{d}x}{x^2 + \frac{9}{4}} = \frac{1}{2} \int_0^{\infty} \frac{\mathrm{d}x}{x^2 + \left(\frac{3}{2}\right)^2} = \frac{1}{2} \lim_{b \to \infty} \int_0^b \frac{\mathrm{d}x}{x^2 + \left(\frac{3}{2}\right)^2}$$

$$= \frac{1}{2} \lim_{b \to \infty} \left[\frac{2}{3} \tan^{-1} \left(\frac{2x}{3}\right)\right]_0^b = \frac{1}{2} \lim_{b \to \infty} \left[\frac{2}{3} \tan^{-1} \left(\frac{2b}{3}\right) - \frac{2}{3} \tan^{-1}(0)\right]$$

$$= \frac{1}{2} \left[\left(\frac{2}{3} \frac{\pi}{2}\right) - 0\right] = \boxed{\frac{\pi}{6}}$$

p.573, pr.18

2. (a) [10 Points] Suppose $a_n = \left(\frac{n-5}{n}\right)^n$. If it converges, find the limit of the sequence $\{a_n\}_{n=1}^{\infty}$.

○ Converges. ○ Diverges. Limit's value = ____________________

Solution: The sequence converges to e^{-5}, since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n-5}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{-5}{n}\right)^n = \boxed{e^{-5}}$$

p.491, pr.86

(b) [10 Points] Investigate the convergence/divergence of the series $\sum_{n=1}^{\infty}\frac{1+\cos n}{n^2}$.

○ Converges. ○ Diverges.

Solution: Since for every $n \geq 2$,

$$0 < a_n := \frac{1+\cos n}{n^2} \leq \frac{2}{n^2}$$

and the p-series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, by the Direct Comparison Test, the given series converges.

p.695, pr.37

(c) [15 Points] Find the radius and interval of convergence of $\sum_{n=0}^{\infty}\frac{(x+4)^n}{n3^n}$.

Radius of Convergence: ____________________. Interval of Convergence: ____________________

Solution: Let $u_n = \frac{(x+4)^n}{n3^n}$. Then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{\frac{(x+4)^{n+1}}{(n+1)3^{n+1}}}{\frac{(x+4)^n}{n3^n}}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x+4)}{3}\,\frac{n}{n+1}\right| < 1 \Rightarrow \frac{|x+4|}{3}\underbrace{\lim_{n\to\infty}\left(\frac{n}{n+1}\right)}_{=1} < 1$$

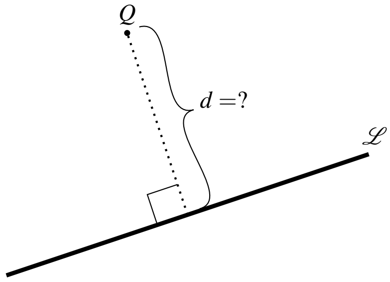
$$\Rightarrow |x+4| < 3$$
$$\Rightarrow -3 < x+4 < 3$$
$$\Rightarrow -7 < x < -1$$

When $x=-7$, we have $\sum_{n=1}^{\infty}\frac{n(-5)^n}{5^n} = \sum_{n=1}^{\infty}\frac{(-1)^n3^n}{n\cdot 3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, the alternating harmonic series and so converges.
When $x=-1$, we have $\sum_{n=1}^{\infty}\frac{3^n}{n3^n} = \sum_{n=1}^{\infty}\frac{1}{n}$, the divergent harmonic series. So the radius of convergence is $R=3$; the interval of convergence is $-7 \leq x < -1$.

p.583, pr.17

3. Given the point $Q(1,-1,5)$ and the line $\mathscr{L}:\begin{cases} x = 1+2t, \\ y = -1+3t, \\ z = 4+t \end{cases}$.

(a) 10 Points Find the distance from the point Q to the line $\mathscr{L}$.



Solution: We shall use the distance formula $d = \dfrac{|\vec{PQ}\times\mathbf{v}|}{|\mathbf{v}|}$. Here (by letting $t=0$) $P(1,-1,4)$ is a point on $\mathscr{L}$ and $\mathbf{v} = 2\mathbf{i}+3\mathbf{j}+\mathbf{k}$ is a vector that is parallel to $\mathscr{L}$. Now we have $\vec{PQ} = 0\mathbf{i}+0\mathbf{j}+\mathbf{k}$ and so

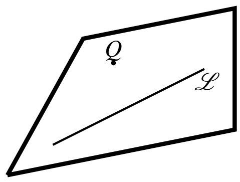
$$\vec{PQ}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 2 & 3 & 1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix}$$
$$= -3\mathbf{i}+2\mathbf{j}+0\mathbf{k}$$

Therefore, we have

$$d = \frac{|\vec{PQ}\times\mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{(-3)^2+(2)^2+(0)^2}}{\sqrt{(2)^2+(3)^2+(1)^2}} = \boxed{\frac{\sqrt{13}}{\sqrt{14}}}$$

p.695, pr.37

(b) 11 Points Find the equation of the plane which contains both the point Q and the line $\mathscr{L}$.



Solution: We know from part (a) that the points $P(1,-1,4)$ and $Q(1,-1,5)$ are on the plane. Setting $t=1$, we get another point $R(3,2,5)$ which must also lie on the plane. Let $\mathbf{a}$ be the vector from $R(3,2,5)$ to $P(1,-1,4)$;

$$\mathbf{a} = (1-3)\mathbf{i}+(-1-2)\mathbf{j}+(4-5)\mathbf{k} = -2\mathbf{i}-3\mathbf{j}-\mathbf{k}.$$

Let $\mathbf{b}$ be the vector from $R(3,2,5)$ to $Q(1,-1,5)$;

$$\mathbf{b} = (1-3)\mathbf{i}+(-1-2)\mathbf{j}+(5-5)\mathbf{k} = -2\mathbf{i}-3\mathbf{j}+0\mathbf{k}.$$

A normal vector $\mathbf{n}$ for the plane may be found by means of cross products.

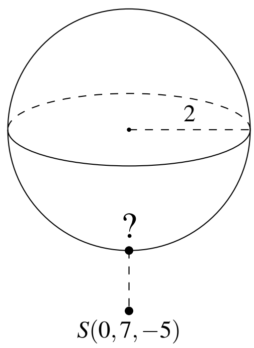
$$\mathbf{n} = \mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -3 & -1 \\ -2 & -3 & 0 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -3 & -1 \\ -3 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -2 & -1 \\ -2 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -2 & -3 \\ -2 & -3 \end{vmatrix}$$
$$= -3\mathbf{i}+2\mathbf{j}-0\mathbf{k}$$

The general equation of a plane, as we know, is:

$$\underbrace{(-3\mathbf{i}+2\mathbf{j}-0\mathbf{k})}_{\mathbf{n}}\cdot\underbrace{((x-1)\mathbf{i}+(y+1)\mathbf{j}+(z-5)\mathbf{k})} = 0$$
$$\Rightarrow 3(x-1)-2(y+1)-0(z-5) = 0$$
$$\Rightarrow \boxed{3x-2y=5}$$

p.695, pr.37

(c) 12 Points Find the point on the sphere $x^2+(y-3)^2+(z+5)^2 = 4$ nearest to the point $S(0,7,-5)$.



Solution: Both the center $(0,3,-5)$ and the point $(0,7,-5)$ lie in the plane $z=-5$, so the point on the sphere nearest to $(0,7,-5)$ should also be in the same plane. In fact it should lie on the line segment between $(0,3,-5)$ and $(0,7,-5)$, thus the point occurs when $x=0$ and $z=- \Rightarrow 0^2+(y-3)^2+(-5+5)^2 = 4 \Rightarrow x^2+(y-3)^2+(z+5)^2 = 4y = 3\pm 2 \Rightarrow (05,-5)$.

p.583, pr.17