Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- Calculators, cell phones off and away!.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 90 min.

Do not write in the table to the right.

Problem	Points	Score
1	32	
2	35	
3	33	
Total:	100	

1. (a) [10 Points] Find the value of $\displaystyle\int_0^{\pi/2} \frac{\cos x\, dx}{\sqrt{1+\sin^2 x}}$.

Solution: Let $u = \sin x$. Then $du = \cos x\, dx$. When $x = 0$, we have $u = \sin(0) = 0$ and when $x = \pi/2$, we have $u = \sin(\pi/2) = 1$. Therefore, using $\sinh^{-1} x = \ln(x+\sqrt{x^2+1})$, we have

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos x\, dx}{\sqrt{1+\sin^2 x}} = \int_0^1 \frac{du}{\sqrt{1+u^2}} &= \left[\sinh^{-1} u\right]_0^1 = \sinh^{-1}(1) - \sinh^{-1}(0)\\ &= \ln(1+\sqrt{1^2+1}) - \ln(0+\sqrt{0^2+1}) = \ln(1+\sqrt{1+1}) - \ln(1)\\ &= \ln(1+\sqrt{2}) - \ln(1) = \ln(1+\sqrt{2}) - 0 = \boxed{\ln(1+\sqrt{2})}\end{aligned}$$

p.652, pr.3

(b) [12 Points] Find the integral $\displaystyle\int \frac{d\theta}{\theta^4+5\theta^2+4}$.

Solution: Appropriate partial fraction decomposition is

$$\begin{aligned}\frac{1}{\theta^4+5\theta^2+4} &= \frac{1}{(\theta^2+4)(\theta^2+1)} = \frac{A\theta+B}{\theta^2+4} + \frac{C\theta+D}{\theta^2+1} \Rightarrow 1 = (A\theta+B)(\theta^2+1) + (C\theta+D)(\theta^2+4)\\ &\Rightarrow 1 = (A+C)\theta^3 + (B+D)\theta^2 + (A+4C)\theta + B + 4D\\ &\Rightarrow A+C = 0,\ \ B+D = 0,\ \ A+4C = 0,\ \ B+4D = 1\\ &\Rightarrow C = -A,\ \ D = -B,\ \ A+4(-A) = 0 \Rightarrow -3A = 0 \Rightarrow A = 0 \Rightarrow C = 0\\ &\Rightarrow B+4(-B) = 1 \ \ \Rightarrow B = -1/3 \ \ \Rightarrow D = 1/3\\ &\Rightarrow \frac{1}{\theta^4+5\theta^2+4} = \frac{-1/3}{\theta^2+4} + \frac{1/3}{\theta^2+1} \Rightarrow \int \frac{d\theta}{\theta^4+5\theta^2+4} = \int \frac{-1/3}{\theta^2+4} d\theta + \int \frac{1/3}{\theta^2+1} d\theta\\ &\Rightarrow \int \frac{d\theta}{\theta^4+5\theta^2+4} = -\frac{1}{12}\int \frac{1}{(\theta/2)^2+1} d\theta + \frac{1}{3}\int \frac{1}{\theta^2+1} d\theta = \boxed{-\frac{1}{6}\tan^{-1}(\theta/2) + \frac{1}{3}\tan^{-1}(\theta) + K}\end{aligned}$$

We have

p.573, pr.18

(c) [10 Points] Find the integral $\displaystyle\int \sec^4 x\, \tan^2 x\, dx$.

Solution: Let $u = \tan x$ and so $du = \sec^2 x\, dx$. Then we have

$$\begin{aligned}\int \sec^4 x\, \tan^2 x\, dx = \int \sec^2 x\, \tan^2 x \sec^2 x\, dx &= \int \underbrace{(1+\tan^2 x)}_{1+u^2}\ \underbrace{\tan^2 x}_{u^2}\ \underbrace{\sec^2 x\, dx}_{du}\\ &= \int (1+u^2)u^2\, du = \int (u^2+u^4)\, du\\ &= \frac{1}{3}u^3 + \frac{1}{5}u^5 + c = \boxed{\frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + c}\end{aligned}$$

p.573, pr.18

2. (a) 14 Points Investigate the convergence or divergence of $\int_{-\infty}^{\infty} \frac{2\,dx}{e^x + e^{-x}}$.

○ Converges. ○ Diverges. Integral's value = ____________

Solution: Since the integrand is even function, $\int_{-\infty}^{\infty} \frac{2\,dx}{e^x + e^{-x}} = 2\int_{0}^{\infty} \frac{2\,dx}{e^x + e^{-x}}$. Now since $0 < \frac{4}{e^x + e^{-x}} < \frac{4}{e^x}$ for $x > 0$ and

$$\int_0^\infty \frac{4\,dx}{e^x} = \lim_{b\to\infty}\int_0^b \frac{4\,dx}{e^x} = 4\lim_{b\to\infty}\int_0^b e^{-x}\,dx = 4\lim_{b\to\infty}\left[\frac{e^{-x}}{-1}\right]_0^b = 4\lim_{b\to\infty}\left(-e^{-b}+1\right) = 4(0+1) = 4$$

is a convergent integral, $\int_0^\infty \frac{4\,dx}{e^x + e^{-x}}$ converges by the Direct Comparison Test.
If we let $y = e^x$, then we have $dy = e^x\ dx$. Therefore

$$\int \frac{4}{e^x + e^{-x}}\ dx = \int \frac{4}{y^2+1}\ dy = 4\tan^{-1} y + C = 4\tan^{-1} e^x + C$$

Then we have

$$\int_0^\infty \frac{4\,dx}{e^x + e^{-x}} = \lim_{b\to\infty}\int_0^b \frac{4}{e^x + e^{-x}}\ dx = \lim_{b\to\infty}\left[4\tan^{-1} e^x\right]_0^b = \lim_{b\to\infty}\left[4\left(\tan^{-1} e^b\right) - 4\left(\tan^{-1} e^0\right)\right]$$
$$= 4\frac{\pi}{2} - 4\frac{\pi}{4} = \boxed{\pi}$$

Therefore the improper integral *converges* and has value π.

p.491, pr.86

(b) 11 Points Investigate the convergence or divergence of $\{a_n\}$ if $a_n = n\left(1 - \cos\frac{1}{n}\right)$. If it converges, find its limit.

○ Converges. ○ Diverges. Limit's Value: ____________

Solution: Let $x = \frac{1}{n}$. Then as $n \to \infty$, $x \to 0$. Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\left(1 - \cos\frac{1}{n}\right) = \lim_{n\to\infty} \frac{1-\cos\frac{1}{n}}{1/n} = \lim_{x\to 0}\frac{1-\cos x}{x} \overset{(L'H)}{=} \lim_{x\to 0}\frac{\sin x}{1} = \lim_{x\to 0}\sin x$$
$$= \sin(0) = \boxed{0}$$

The sequence *converges* to 0.

p.695, pr.37

(c) 10 Points Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$.

Solution:

$$s_k = \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) + \cdots + \left(\frac{1}{\sqrt{k-1}} - \frac{1}{\sqrt{k}}\right) + \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right)$$
$$= 1 - \frac{1}{\sqrt{k+1}}$$

Hence the formula is $\boxed{s_n = 1 - \frac{1}{\sqrt{n+1}}}$. Therefore $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \frac{1}{\sqrt{n+1}}\right) = 1$. Hence the series *converges* and has sum 1.

p.695, pr.37

3\. (a) 11 Points Does the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converge absolutely, conditionally, or diverge? Justify your answer.

○ Converges absolutely. ○ Converges conditionally. ○ Diverges. Test Used: ____________

Solution: This is an alternatin series of the form $\sum_{n=1}^{\infty}(-1)^n u_n$ where $u_n = \frac{1}{\sqrt{n}}$. Then since

- $\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} > 0$ for all n,
- $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

the series converges by Alternating Series Test. But since

$$\sum_{n=1}^{\infty} |(-1)^n u_n| = \sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

is a divergent p-series ($p = 1/2 < 1$), the given *series converges conditionally*.

p.573, pr.36

(b) 10 Points Investigate the convergence or divergence of $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n\,3^{n-1}}$.

○ Converges. ○ Diverges. Test Used: ____________

Solution: Let $u_n = \frac{2^{n+1}}{n\,3^{n-1}} > 0$ for all n. Then $u_{n+1} = \frac{2^{n+2}}{(n+1)\,3^n}$ using Ratio Test, we have.

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2^{n+2}}{(n+1)\,3^n}\,\frac{n\,3^{n-1}}{2^{n+1}}$$

$$= \lim_{n\to\infty} \frac{2\cdot \cancel{2^{n+1}}}{(n+1)\,3\cdot\cancel{3^{n-1}}}\,\frac{n\,\cancel{3^{n-1}}}{\cancel{2^{n+1}}} = \underbrace{\lim_{n\to\infty}\left(\frac{2n}{3(n+1)}\right)}_{=2/3}$$

$$\Rightarrow \rho = 2/3 < 1$$

Series converges by Ratio Test.

p.112, pr.26

(c) 12 Points Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{(n!)^2}{2^n\,(2n)!}x^n$.

Radius of Convergence: ____________.

Solution: Let $u_n = \frac{(n!)^2}{2^n\,(2n)!}x^n$. Then $u_{n+1} = \frac{((n+1)!)^2}{2^{n+1}\,(2n+2)!}x^{n+1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{((n+1)!)^2 x^{n+1}}{2^{n+1}\,(2(n+1))!}\,\frac{2^n(2n)!}{(n!)^2 x^n}\right| < 1$$

$$\Rightarrow \frac{|x|}{2}\lim_{n\to\infty}\left|\frac{(n+1)^2}{(2n+2)(2n+1)}\right| < 1 \Rightarrow \frac{|x|}{2}\underbrace{\lim_{n\to\infty}\left(\frac{(n+1)^2}{(2n+2)(2n+1)}\right)}_{=1/4} < 1$$

$$\Rightarrow |x| < 8 \Rightarrow -8 < x < 8 \Rightarrow \boxed{R = 8}$$

p.583, pr.17