Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- Calculators, cell phones off and away!.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 70 min.

Do not write in the table to the right.

Problem	Points	Score
1	15	
2	27	
3	25	
4	33	
Total:	100	

1. 15 Points Find the radius and interval of convergence of $\sum_{n=0}^{\infty} \frac{(3x+1)^{n+1}}{2n+2}$.

Radius of Convergence: ______________. Interval of Convergence: ______________

Solution: Let $u_n = \frac{(3x+1)^{n+1}}{2n+2}$. Then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{\frac{(3x+1)^{n+2}}{2n+4}}{\frac{(3x+1)^{n+1}}{2n+2}}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(3x+1)^{n+2}}{2n+4}\,\frac{2n+2}{(3x+1)^{n+1}}\right| < 1 \Rightarrow |3x+1|\underbrace{\lim_{n\to\infty}\left(\frac{2n+2}{2n+4}\right)}_{=1} < 1$$

$$\Rightarrow |3x+1| < 1$$
$$\Rightarrow -1 < 3x+1 < 1$$
$$\Rightarrow \frac{-2}{3} < x < 0$$

When $x = -\frac{2}{3}$, we have $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+2}$, alternating series and converges by Alternating Series Test.

When $x = 0$, we have $\sum_{n=1}^{\infty} \frac{1^{n+1}}{2n+2} = \sum_{n=1}^{\infty} \frac{1}{2n+2}$, divergent series by Comparison Test with $\sum \frac{1}{n}$.

So the radius of convergence is $R = \frac{1}{3}$; the interval of convergence is $-\frac{2}{3} \le x \le 0$.

p.583, pr.17

2. (a) [10 Points] Suppose $a_n = \left(\frac{3n+1}{3n-1}\right)^n$. If it converges, find the limit of the sequence $\{a_n\}_{n=1}^{\infty}$.

○ Converges. ○ Diverges. Limit's value = ______________

Solution: The sequence converges to e^{-5}, since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{3n+1}{3n-1}\right)^n = \lim_{x\to\infty}\exp\left(x\ln\frac{3x+1}{3x-1}\right) = \exp\left(\lim_{x\to\infty}\frac{\ln(3x+1)-\ln(3x-1)}{\frac{1}{x}}\right)$$

$$= \exp\left(\lim_{x\to\infty}\frac{\frac{3}{3x+1}-\frac{3}{3x-1}}{-\frac{1}{x^2}}\right) = \exp\left(\lim_{x\to\infty}\frac{6x^2}{(3x+1)(3x-1)}\right) = \exp\left(\lim_{x\to\infty}\frac{6}{9-1/x^2}\right) = \boxed{e^{2/3}}$$

p.491, pr.86

(b) [6 Points] Investigate the convergence/divergence of the series $\sum_{n=1}^{\infty}\left(\frac{3n+1}{3n-1}\right)^n$.

○ Converges. ○ Diverges.

Solution: Thes series diverges by the nth Term Test, since $\lim_{n\to\infty\to\infty} a_n = e^{2/3} \neq 0$.

p.695, pr.37

(c) [11 Points] Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}} - \frac{(-1)^n}{5^n}\right)$.

○ Converges. ○ Diverges. Series' Sum = ______________

Solution: This is the difference of two convergent geometric series $\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}$ and $\sum_{n=1}^{\infty}\frac{(-1)^n}{5^n}$. The first series is of the form $\sum_{n=1}^{\infty} ar^{n-1}$ where $a=1$ and $r=1/2$ and so has the sum $\frac{a}{1-r} = \frac{1}{1-1/2} = 2$. Similarly, the second series is of the form $\sum_{n=1}^{\infty} ar^{n-1}$ where $a=-1/5$ and $r=-1/5$ and so has the sum $\frac{a}{1-r} = \frac{-1/5}{1+1/5} = -1/6$. Finally the sum of given series is $\sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}} - \frac{(-1)^n}{5^n}\right) = 2-(-1/6) = \boxed{\frac{13}{6}}$

p.551, pr.13

3. (a) 14 Points Investigate the series for absolute or conditional convergence or divergence: $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\ln n}$.

○ Converges. ○ Diverges. Test Used: ____________

Solution: converges by the Alternating Series Test because

$$\begin{cases} f(x) = \ln x \text{ is is an increasing function of } x \Rightarrow \frac{1}{\ln x} \text{ is decrasing} \\ \Rightarrow u_n \geq u_{n+1} \text{ for } n \geq 1; \\ \text{also } u_n \geq 0 \text{ for } n \geq 1; \\ \text{and } \lim_{n\to\infty} \frac{1}{\ln n} = 0 \end{cases}$$

However the convergence is conditional, since $\sum_{n=1}^{\infty}\left|(-1)^n \frac{1}{\ln n}\right| = \sum_{n=1}^{\infty} \frac{1}{\ln n}$ diverges by Ordinary Comparison Test $\frac{1}{\ln n} > \frac{1}{n} > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ is harmonic series which diverges.

p.695, pr.37

(b) 11 Points Investigate the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$.

○ Converges. ○ Diverges. Test Used: ____________

Solution: First notice for all $n \geq 1$, that

$$0 < a_n := \frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series with $p = 2 > 1$, it follows by Comparison Test that the original series converges.

p.695, pr.37

4. Evaluate the following integrals.

(a) 11 Points $\displaystyle\int \frac{e^t}{e^{2t}+3e^t+2}\,\mathrm{d}t = ?$ (First use a substitution.)

Solution: Let $x = e^t$ and so $dx = e^t\,\mathrm{d}t$. Then, we have

$$\int \frac{e^t}{e^{2t}+3e^t+2}\,\mathrm{d}t = \int \frac{1}{x^2+3x+2}\,\mathrm{d}x = \int \frac{1}{(x+1)(x+2)}\,\mathrm{d}x = \int \left(\frac{1}{x+1} - \frac{1}{x+2}\right)\mathrm{d}x$$
$$= \ln|x+1| - \ln|x+2| + C$$
$$= \ln\left|\frac{x+1}{x+2}\right|$$
$$= \boxed{\ln\left(\frac{e^t+1}{e^t+2}\right) + C}$$

p.652, pr.2

(b) 12 Points $\displaystyle\int \sin^2\theta\cos^5\theta\,\,\mathrm{d}\theta = ?$

Solution: Let $u = \sin\theta$ and so $du = \cos\theta\,\mathrm{d}\theta$. Then

$$\int \sin^2\theta\cos^5\theta\,\,\mathrm{d}\theta = \int \sin^2\theta\cos^4\theta\cos\theta\,\mathrm{d}\theta = \int \sin^2\theta(1-\sin^2\theta)^2\cos\theta\,\,\mathrm{d}\theta$$
$$= \int u^2(1-u^2)^2\,\,\mathrm{d}u = \int u^2(1-2u^2+u^4)\,\,\mathrm{d}u = \int (u^2-2u^4+u^6)\,\,\mathrm{d}u$$
$$= \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} + c$$
$$= \boxed{\frac{1}{3}\sin^3\theta - \frac{2\sin^5\theta}{5} + \frac{1}{7}\sin^7\theta + c}$$

p.573, pr.38

(c) 10 Points $\displaystyle\int_{-\infty}^{0} xe^{3x}\,\mathrm{d}x =$

○ Converges. ○ Diverges. Integral's value = ____________________

Solution: Let $u = x$ and so $dv = e^{3x}\,\mathrm{d}x$. Then $du = dx$ and choose $v = \frac{1}{3}e^{3x}$. Integrating by parts, we have

$$\int xe^{3x}\,\mathrm{d}x = \int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u = \frac{x}{3}e^{3x} - \int \frac{1}{3}e^{3x}\,\mathrm{d}x = \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + C.$$

Then evaluate the improper integral:

$$\int_{-\infty}^{0} xe^{3x}\,dx = \lim_{b\to-\infty}\int_{b}^{0} xe^{3x}\,dx$$
$$= \lim_{b\to-\infty}\left[\frac{x}{3}\mathrm{e}^{3x} - \frac{1}{9}\mathrm{e}^{3x}\right]_b^0 = \lim_{b\to-\infty}\left[-\frac{1}{9} - \left(\frac{b}{3}\mathrm{e}^{3b} - \frac{1}{9}\mathrm{e}^{3b}\right)\right]$$
$$= -\frac{1}{9} - 0 = \boxed{-\frac{1}{9}}$$

p.573, pr.18