# OKAN UNIVERSITY

## Department of Mathematics

**Math 112**
**Calculus for Engineering II**
**Second Exam**
**SOLUTIONS**

April 13, 2012
15:00-16:15

Surname : ______________________
Name : ______________________
ID # : ______________________
Department : ______________________
Section : ______________________
Instructor : ______________________
Signature : ______________________

- The exam consists of 5 questions
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

*GOOD LUCK!*

Please do not write below this line.

| Q1 | Q2 | Q3 | Q4 | Q5 | TOTAL |
|---|---|---|---|---|---|
| | | | | | |
| 20 | 15 | 15 | 20 | 30 | 100 |

**1.** (20pts) Find the area of the surface generated by revolving the curve $y = \left(4 - x^{2/3}\right)^{3/2}$ between $x = 1$ and $x = 8$ around y-axis.

**SOLUTION:**

**(Method 1)**

$$\begin{aligned}
f(x) &= \left(4 - x^{2/3}\right)^{3/2} \\
f'(x) &= \frac{3}{2}\left(4 - x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) \\
&= -x^{-1/3}\left(4 - x^{2/3}\right)^{1/2} \\
L &= \int_1^8 \sqrt{1 + \left[f'(x)\right]^2}\, dx \\
A &= 2\pi \int_1^8 x\sqrt{1 + \left[-x^{-1/3}\left(4 - x^{2/3}\right)^{1/2}\right]^2}\, dx \\
&= 2\pi \int_1^8 x\sqrt{4x^{-2/3}}\, dx = 4\pi \int_1^8 x^{2/3}\, dx \\
&= 4\pi \left[\frac{3}{5}x^{5/3}\right]_1^8 = 4\pi\left(\frac{96}{5} - \frac{3}{5}\right) = \frac{372}{5}\pi units.
\end{aligned}$$

**(Method 2)**

First, we express the in the form $x = g(y) \geq 0$, where $y^{2/3} = 4 - x^{2/3} \Longleftrightarrow x^{2/3} = 4 - y^{2/3} \Longleftrightarrow g(y) = \left(4 - y^{2/3}\right)^{3/2}$ where $x = 8 \Longrightarrow y = 0$ and $x = 1 \Longrightarrow y = 3^{3/2}$

Then $g'(y) = -\frac{3}{2}\left(4 - y^{2/3}\right)^{1/2} \cdot \frac{2}{3}y^{-1/3}$. Thus $1 + \left[g'(y)\right]^2 = 1 + \left[-\frac{3}{2}\left(4 - y^{2/3}\right)^{1/2} \cdot \frac{2}{3}y^{-1/3}\right]^2 = 1 + \left(4 - y^{2/3}\right)y^{-2/3} = 1 + 4y^{-2/3} - 1 = 4y^{-2/3}$.

Therefore, $2\pi g(y)\sqrt{1 + \left(g'(y)\right)^2} = 2\pi\left(4 - y^{2/3}\right)^{3/2}\sqrt{4y^{-2/3}} = 4\pi\left(4 - y^{2/3}\right)^{3/2} y^{-1/3}$

$$\begin{aligned}
S &= \int_c^d 2\pi g(y)\sqrt{1 + \left(g'(y)\right)^2}\, dy \\
&= \int_0^{\sqrt{27}} 4\pi\left(4 - y^{2/3}\right)^{3/2} y^{-1/3}\, dy.
\end{aligned}$$

To evaluate the last integral, we substitute $u = 4 - y^{2/3}$ and so $du = -\frac{2}{3}y^{2/3-1}\, dy = -\frac{2}{3}y^{-1/3}\, dy$. When $y = 0$, we have $u = 4$ and $y = \sqrt{27}$, we have $u = 1$.

Therefore

$$\begin{aligned}
S &= -\frac{3}{2}\int_4^1 4\pi\left(4 - y^{2/3}\right)^{3/2}\left(-\frac{2}{3}\right) y^{-1/3}\, dy \\
&= \frac{3}{2}4\pi \int_1^4 u^{3/2}\, du \\
&= \frac{372}{5}\pi units^2
\end{aligned}$$

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**2.** (15pts) Suppose $f(x) = x^5 + 2x^3 + 4x$, and $g(x) = f^{-1}(x)$ for $-\infty < x < \infty$.
**(a)** Evaluate $f(1)$ and $g(7)$.
**(b)** Evaluate $g'(7)$.
**SOLUTION:**
**(a)**
Clearly, $f(1) = (1)^5 + 2(1)^3 + 4(1) = 7$ and so $g(7) = f^{-1}(7) = 1$.
**(b)**
By the Inverse Function Theorem, we have

$$g'(7) = \frac{1}{f'(1)} = \left[\frac{1}{5x^4 + 6x^2 + 4}\right]_{x=1} = \frac{1}{5(1)^4 + 6(1)^2 + 4} = \frac{1}{15}.$$

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**3.** (15pts) Find the limit, if it exists

$$\lim_{x\to\infty}\left(1+\sin\frac{3}{x}\right)^{x}.$$

**SOLUTION:**
First, the limit has the form $1^{\infty}$.
Let $y=\left(1+\sin\dfrac{3}{x}\right)^{x}$.
Then

$$\begin{aligned}\ln y &= x\ln\left(1+\sin\frac{3}{x}\right)\\ &= \frac{\ln\left(1+\sin\frac{3}{x}\right)}{\frac{1}{x}}\end{aligned}$$

Now the limit of this function has the form $\dfrac{0}{0}$.
Thus

$$\begin{aligned}\lim_{x\to\infty}\ln y &= \lim_{x\to\infty}\frac{\ln\left(1+\sin\frac{3}{x}\right)}{\frac{1}{x}}\\ &= \lim_{x\to\infty}\frac{\frac{1}{1+\sin\frac{3}{x}}\left(-\frac{3}{x^2}\right)\cos\frac{3}{x}}{-\frac{1}{x^2}} \quad \text{(L'Hopital's Rule)}\\ &= \lim_{x\to\infty}\frac{3}{1+\sin\frac{3}{x}}\cos\frac{3}{x}=3.\end{aligned}$$

Therefore

$$\ln y\to 3 \ \text{ and so } \ y\to e^{3}.$$

Thus

$$\lim_{x\to\infty}\left(1+\sin\frac{3}{x}\right)^{x}=e^{3}$$

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**4.** (20pts) Suppose $\cosh x = 2$.
**(a)** Find all values of $x$ satisfying the given equation.
**(b)** Find $\cosh 2x$.
**SOLUTION:**
**(a)**
We have

$$\begin{aligned}
\cosh x &= 2 \Longleftrightarrow \frac{e^x + e^{-x}}{2} = 2 \Longleftrightarrow e^{2x} + 1 = 4e^x \\
&\Longleftrightarrow w^2 - 4w + 1 = 0, \quad \text{where } w = e^x. \\
&\Longleftrightarrow w_{1,2} = 2 \pm \sqrt{3} \\
&\Longleftrightarrow e^x = 2 \pm \sqrt{3} \\
&\Longleftrightarrow x = \ln\left(2 \pm \sqrt{3}\right)
\end{aligned}$$

**(b)**
We may proceed in two ways:
First, from $\cosh x = 2$ and $\cosh^2 x - \sinh^2 x = 1$, we have $(2)^2 - \sinh^2 x = 1 \Longrightarrow \sinh^2 x = 3 \Longrightarrow \sinh x = \pm\sqrt{3}$.
Now we use the identity $\cosh(2x) = \cosh^2 x + \sinh^2 x = 2^2 + \left(\pm\sqrt{3}\right)^2 = 7$.
Alternatively, from the definitions of $\cosh x$ and $\sinh x$, we have

$$\begin{aligned}
\cosh 2x &= \frac{e^{2x} + e^{-2x}}{2}. \\
\text{When } x &= \ln\left(2+\sqrt{3}\right), \ \text{ we have } \ e^x = 2 + \sqrt{3} \Longrightarrow \frac{1}{2}\left(e^{2x} + e^{-2x}\right) = \frac{1}{2}\left[\left(2+\sqrt{3}\right)^2 + \left(2+\sqrt{3}\right)^{-2}\right] \\
\cosh 2x &= \frac{1}{2}\left(7 + 4\sqrt{3} + \frac{1}{7+4\sqrt{3}}\right) = \frac{1}{2}\left(7 + 4\sqrt{3} + \frac{7-4\sqrt{3}}{49-48}\right) = 7 \\
\text{When } x &= \ln\left(2-\sqrt{3}\right) \Longrightarrow e^x = 2 - \sqrt{3} \\
e^{2x} &= \left(2-\sqrt{3}\right)^2 = 7 - 4\sqrt{3} \ \text{ and } \ e^{-2x} = \left(2-\sqrt{3}\right)^{-2} = \frac{1}{7-4\sqrt{3}} = 7 + 4\sqrt{3} \\
\cosh 2x &= \frac{1}{2}\left(e^{2x} + e^{-2x}\right) = \frac{1}{2}\left(7 - 4\sqrt{3} + 7 + 4\sqrt{3}\right) = 7
\end{aligned}$$

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**5.** (30pts) Evaluate the following integrals. (Remember $\arctan x = \tan^{-1} x$)

$$\textbf{(a)} \int x \arctan x \; dx, \qquad \textbf{(b)} \int \frac{dx}{\sqrt{x} + 4x\sqrt{x}}, \qquad \textbf{(c)} \int \sec^3 x \tan^3 x \; dx$$

**SOLUTION:**

**(a)** We integrate by parts.

Let

$$u = \tan^{-1} x \qquad dv = x \; dx$$
$$du = \tfrac{1}{1+x^2} \; dx \qquad v = \tfrac{1}{2}x^2$$

Then

$$\begin{aligned}
\int x \tan^{-1} x \; dx &= \int u \; dv \\
&= uv - \int v du \\
&= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2} \; dx = \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2} \; dx \\
&= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) \; dx \\
&= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\left(x - \tan^{-1} x\right) + C.
\end{aligned}$$

**(b)**

First, we substitute

$$y = \sqrt{x} \;\text{ and so }\; x = y^2 \;\text{ and }\; dx = 2y \; dy.$$

Then we have

$$\begin{aligned}
\int \frac{dx}{\sqrt{x} + 4x\sqrt{x}} &= \int \frac{2y \; dy}{y + 4y^2 y} = 2\int \frac{dy}{1+4y^2} = \int \frac{2 \; dy}{1 + (2y)^2} = \tan^{-1}(2y) + C \\
&= \tan^{-1}\left(2\sqrt{x}\right) + C.
\end{aligned}$$

**(c)**

We have the product of odd powers of tangent and secant. Thus

$$\begin{aligned}
\int \sec^3 x \tan^3 x \; dx &= \int \sec^2 x \tan^2 x \sec x \tan x \; dx \\
&= \int \sec^2 x \left(\sec^2 x - 1\right) \sec x \tan x \; dx \qquad y = \sec x \text{ and } dy = \sec x \tan x \; dx \\
&= \int y^2 \left(y^2 - 1\right) \; dy \\
&= \int \left(y^4 - y^2\right) \; dy = \frac{1}{5}y^5 - \frac{1}{3}y^3 + C = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C.
\end{aligned}$$

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