Math 114 Summer 2018 **Second Exam** July 25, 2018

Your Name / Ad - Soyad (**75 min.**) Signature / İmza

Student ID # / Öğrenci No (**mavi tükenmez!**)

Problem	1	2	3	4	Total
Points:	24	25	27	24	100
Score:					

Time limit is **75 minutes**. If you need more room on your exam paper, you may use the empty spaces on the paper. You have to show your. Answers without reasonable-even if your results are true- work will either get zero or very little credit.

1. (a) (12 Points) Investigate the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$.

○ Converges. ○ Diverges. Test Used: ______________

Solution: Use the Ratio Test. Let $a_n = \frac{n!}{(2n+1)!} > 0$ for each $n \geq 1$. We have

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!}{(2n+3)!}\frac{(2n+1)!}{(n)!} = \lim_{n\to\infty}\frac{(2n+2)^{1}(2n+1)!\,n!}{(2n+3)(2n+2)_{2}(2n+1)!}\frac{(2n+1)!}{(n)!}$$

$$= \lim_{n\to\infty}\frac{1}{4n+6} = \boxed{0<1}$$

The series converges by Ratio Test.

p.72, pr.8

Justify your answer.

(b) (12 Points) Investigate the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{n-1}{n^4+2}$.

○ Converges. ○ Diverges. Test Used: ______________

Solution: Use Direct Comparison Test with $\sum_{n=1}^{\infty}\frac{1}{n^3}$ which is a convergent p-series since $p = 3 > 1$. Let $a_n = \frac{n-1}{n^4+2} > 0$ for each $n \geq 1$. For $n \geq 1$, we have

$$n^4 \leq n^4+2 \Rightarrow \frac{1}{n^4} \geq \frac{1}{n^4+2} \Rightarrow \frac{n}{n^4} \geq \frac{n}{n^4+2}$$

$$\Rightarrow \frac{1}{n^3} \geq \frac{n}{n^4+2} \geq \frac{n-1}{n^4+2}$$

By Direct Comparison Test,the given series converges.

p.72, pr.8

Justify your answer.

2. (a) (10 Points) Does the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{2^n}$ converge absolutely? Converge Conditionally? Or diverge? Justify your answer.

○ Converges. ○ Diverges. Test Used: ____________________

Solution: Converges absolutely. Let $a_n = (-1)^{n+1}\frac{3}{2^n}$ for each $n \geq 1$. We have

$$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{3}{2^n}\right| = \sum_{n=1}^{\infty}\frac{3}{2^n} = \sum_{n=1}^{\infty}\frac{3}{2}\left(\frac{1}{2}\right)^{n-1} = \sum_{n=1}^{\infty} a\, r^{n-1}$$

$$\Rightarrow a = \frac{3}{2},\ r = \frac{1}{2} \Rightarrow \boxed{|r| = \frac{1}{2} < 1}$$

This is a convergent geometric series because $|r| < 1$.

p.72, pr.8

Justify your answer.

(b) (15 Points) Find the radius and interval of convergence of $\sum_{n=0}^{\infty}\frac{(3x-2)^n}{n}$.

Radius of Convergence: ____________________. Interval of Convergence: ____________________

Solution: Let $u_n = \frac{(3x-2)^n}{n}$. Then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{\frac{(3x-2)^{n+1}}{n+1}}{\frac{(3x-2)^n}{n}}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|(3x-2)\frac{n+1}{n}\right| < 1 \Rightarrow |(3x-2)|\underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)}_{=1} < 1$$

$$\Rightarrow |3x-2| < 1$$
$$\Rightarrow -1 < 3x-2 < 1 \Rightarrow 1 < 3x < 3$$
$$\Rightarrow \frac{1}{3} < x < 1$$

When $x = \frac{1}{3}$, we have $\sum_{n=1}^{\infty}\frac{(3(\frac{1}{3})-2)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$,a conditionally convergent series.

When $x = 1$, we have $\sum_{n=1}^{\infty}\frac{((3)(1)-2)^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$, a divergent series. So the radius of convergence is $R = 1$; the interval of convergence is $\frac{1}{3} \leq x < 1$.

p.583, pr.17

3. (a) (12 Points) Find the Taylor series generated by $f(x) = \ln x$ about $a = 1$. Use this to evaluate the limit $\lim\limits_{x\to 1} \dfrac{\ln x}{x-1}$. (*Hint*: You may use the series $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots + (-1)^n \dfrac{x^n}{n} + \cdots = \sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n}, \quad -1 < x \leq 1$)

Solution: By replacing x by $x-1$ in the series for $\ln(1+x)$, we have

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{3}(x-1)^3 - \cdots + (-1)^{n-1}\frac{(x-1)^n}{n} + \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}, \;\; 0 < x \leq 2$$

from which we find that

$$\frac{\ln x}{x-1} = 1 - \frac{1}{2}(x-1) + \frac{1}{3}(x-1)^2 - \frac{1}{3}(x-1)^2 - \cdots + (-1)^{n-1}\frac{(x-1)^{n-1}}{n} + \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^{n-1}}{n}, \;\; 0 < x \leq 2.$$

Therefore, the limit is

$$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\left(1 - \frac{1}{2}(x-1) + \frac{1}{3}(x-1)^2 - \frac{1}{3}(x-1)^2 - \cdots + (-1)^{n-1}\frac{(x-1)^{n-1}}{n} + \cdots\right) = \boxed{1}$$

p.72, pr.8

(b) (15 Points) Using the series $\sin x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$, write the first four nonzero terms for the series $\int \sin(x^2)\; \mathrm{d}x$.

Solution: From the series for $\sin x$,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty,$$

we substitute x^2 for x to obtain

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots + (-1)^n\frac{x^{4n+2}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{(2n+1)!}, \quad -\infty < x < \infty.$$

Therefore,

$$\begin{aligned}\int \sin(x^2)\; \mathrm{d}x &= \int \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots + (-1)^n\frac{x^{4n+2}}{(2n+1)!} + \cdots\right) \mathrm{d}x \\ &= \boxed{C + \frac{x^3}{3} - \frac{x^7}{7\times 3!} + \frac{x^{11}}{11\times 5!} - \frac{x^{15}}{15\times 7!} + \cdots}\end{aligned}$$

p.83, pr.52

4. Given the vectors $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = -2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$.

   (a) (12 Points) If the angle between $\mathbf{u}$ and $\mathbf{v}$ is θ, then find $\cos\theta$.

Solution: First the dot product is

$$\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{(2\mathbf{i}+\mathbf{j}+\mathbf{k}) \bullet (2\mathbf{i}+3\mathbf{j}+\mathbf{k})}{|2\mathbf{i}+\mathbf{j}+\mathbf{k}||2\mathbf{i}+3\mathbf{j}+\mathbf{k}|}$$
$$= \frac{(2)(2)+(1)(3)+(1)(1)}{\sqrt{(2)^2+(1)^2+(1)^2}\sqrt{(2)^2+(3)^2+(1)^2}}$$
$$= \frac{4+3+1}{\sqrt{6}\sqrt{14}} = \boxed{\frac{8}{\sqrt{84}}}$$

p.82, pr.35

   (b) (12 Points) Find the volume of the parallelepiped that is spanned by the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Solution: We have $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = -2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$.

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} 2 & 1 & 1 \\ 2 & 3 & 1 \\ -2 & 4 & 1 \end{vmatrix}$$
$$= (2)\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} - (1)\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} + (1)\begin{vmatrix} 2 & 3 \\ -2 & 4 \end{vmatrix}$$
$$= -2 - 4 + 14 = \boxed{8}$$

Hence Volume $= |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = 8$.

p.688, pr.48