Your Name / Ad - Soyad ______ (**70 min.**)

Signature / İmza ______

Student ID # / Öğrenci No ☐☐☐☐☐☐☐☐☐ (**mavi tükenmez!**)

Problem	1	2	3	4	Total
Points:	20	30	30	20	100
Score:					

Time limit is **70 minutes**. If you need more room on your exam paper, you may use the empty spaces on the paper. You have to show your. Answers without reasonable-even if your results are true- work will either get zero or very little credit.

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1. (a) (10 Points) Does the series $\sum_{n=1}^{\infty}\left(\frac{1}{8}\right)^n = \frac{1}{8}+\left(\frac{1}{8}\right)^2+\left(\frac{1}{8}\right)^3+\cdots$ converge? If it converges, find its sum.

○ Converges. ○ Diverges. Series' Sum: ______

Solution: Series is geometric with $r=\frac{1}{8} \Rightarrow |r| = \frac{1}{8} < 1$ and has sum

$$S = \frac{a}{1-r} = \frac{1/8}{1-\frac{1}{8}} = \boxed{\frac{1}{7}}$$

p.72, pr.15

(b) (10 Points) Use the nth Term Test to investigate the divergence of $\sum_{n=1}^{\infty}\frac{n(n+1)}{(n+2)(n+3)}$ or decide that this cannot be used for this series.

Solution: We can use the nth Term Test to conclude that the series diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n(n+1)}{(n+2)(n+3)} = \lim_{n\to\infty}\frac{n^2+n}{n^2+5n+6} = \lim_{n\to\infty}\frac{1+\frac{1}{n}}{1+\frac{5}{n}+\frac{6}{n^2}} = 1 \neq 0.$$

Hence the series *diverges*.

p.94, pr.34

2. (a) (15 Points) Use the Root Test to investigate the convergence of $\sum_{n=1}^{\infty} \frac{7}{(2n+5)^n}$.

○ Converges. ○ Diverges. Test Used: ____________

Solution: Let $u_n = \frac{7}{(2n+5)^n} > 0$. Then $|u_n| = u_n$ for every $n \geq 1$. Root Test gives

$$\rho = \lim_{n\to\infty} \sqrt[n]{|u_n|} = \lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \sqrt[n]{\frac{7}{(2n+5)^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{7}}{2n+5} = 0 < 1$$

The series converges.

p.72, pr.8

(b) (15 Points) Use the Ratio Test to investigate the convergence of $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n3^{n-1}}$.

○ Converges. ○ Diverges. Test Used: ____________

Solution: Let $u_n = \frac{2^{n+1}}{n3^{n-1}} > 0$. Then $|u_n| = u_n$ for every $n \geq 1$. Ratio Test gives

$$\rho = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{2^{n+2}}{(n+1)3^n}}{\frac{2^{n+1}}{n3^{n-1}}} \right| = \lim_{n\to\infty} \left(\frac{2^{n+1}\cdot 2}{(n+1)3^{n-1}\cdot 3} \frac{n\cdot 3^{n-1}}{2^{n+1}} \right)$$
$$= \lim_{n\to\infty} \left(\frac{2n}{3n+3} \right) = \frac{2}{3} < 1.$$

Since $\rho < 1$, by Ratio Test, the series *converges*.

p.83, pr.52

3. (a) (15 Points) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converge absolutely? Converges conditionally? Diverges? Justify your answer.

○ Converges Absolutely. ○ converges Conditionally. ○ diverges Test Used: ________

Solution: This series *converges conditionally* by the Alternating Series Test since

$$\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} > 0 \quad \text{and} \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

But the convergence is not absolute since

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

is a divergent p-series (as $p = \frac{1}{2} \leq 1$).

p.95, pr.68

(b) (15 Points) Find the radius and interval of convergence of $\sum_{n=1}^{\infty} (-1)^n \frac{(x+2)^n}{n}$. For what values of x does the series converge conditionally?

○ Radius of Convergence. ○ Interval of Convergence. ○ diverges Test Used: ________

Solution: Let $u_n = (-1)^n \frac{(x+2)^n}{n}$. Then

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n\to\infty} \left| \frac{(-1)\frac{(x+2)^{n+1}}{n+1}}{\frac{(x+2)^n}{n}} \right| < 1 \Rightarrow \lim_{n\to\infty} \left| (x+2)\frac{n+1}{n} \right| < 1 \Rightarrow |(x+2)| \underbrace{\lim_{n\to\infty} \left(1+\frac{1}{n}\right)}_{=1} < 1$$

$$\begin{aligned} &\Rightarrow |x+2| < 1 \\ &\Rightarrow -1 < x+2 < 1 \\ &\Rightarrow -3 < x < -1 \end{aligned}$$

When $x = -3$, we have $\sum_{n=1}^{\infty} \frac{(-3+2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$,a conditionally convergent series.
When $x = -11$, we have $\sum_{n=1}^{\infty} \frac{(-1+2)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$, a divergent series. So the radius of convergence is $R = 1$; the interval of convergence is $-3 \leq x < -1$. Series converges conditionally at $x = -3$.

p.112, pr.26

4. For $-\pi/2 < x < \pi/2$, the series $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots$ converges to $\tan x$.

   (a) (10 Points) Write the first five terms of the series for $\ln|\sec x|$. For what values of x does the series converge?

Solution: We shall integrate term-by-term.

$$\ln|\sec x| + C = \int \tan x \, \mathrm{d}x$$

$$= \int \left(x + \frac{x^3}{3} + \frac{2x^5}{5} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots \right) \mathrm{d}x = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \frac{31x^{10}}{14,175} + \cdots + C;$$

$$x = 0 \Rightarrow C = 0 \Rightarrow$$

$$\ln|\sec x| = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \frac{31x^{10}}{14,175} + \cdots,$$

converges when $-\pi/2 < x < \pi/2$.

p.112, pr.26

   (b) (10 Points) Write the first five terms of the series for $\sec^2 x$. For what values of x does the series converge?

Solution:

$$\sec^2 x = \frac{d}{dx}\left(x + \frac{x^3}{3} + \frac{2x^5}{5} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots \right)$$

$$= 1 + x^2 + \frac{2x^4}{3} + \frac{17x^7}{45} + \frac{62x^8}{315} + \cdots,$$

converges when $-\pi/2 < x < \pi/2$.

p.82, pr.35