

Your Name / Ad - Soyad

(75 min.)

Signature / İmza

Student ID # / Öğrenci No

(mavi tükenmez!)

Problem	1	2	3	4	Total
Points:	30	20	25	25	100
Score:					

1. Find the following limits.

(a) (10 Points) Let $f(x) = x^3 - 3x^2 - 1$, $x \geq 2$. Find the value of df^{-1}/dx at the point $x = -1 = f(3)$.**Solution:**

$$\frac{df}{dx} = 3x^2 - 6x \Rightarrow \left[\frac{df}{dx} \right]_{x=3} = 3(3)^2 - 6(3) = 9 \Rightarrow \left[\frac{df^{-1}}{dx} \right]_{x=-1} = \left[\frac{1}{\frac{df}{dx}} \right]_{x=3} = \boxed{\frac{1}{9}}$$

p.72, pr.15

(b) (10 Points) Evaluate the integral $\int \frac{3 \sec^2 t}{6 + 3 \tan t} dt$.**Solution:** Let $u = 6 + 3 \tan t$ and so $du = 3 \sec^2 t dt$. Then

$$\int \frac{3 \sec^2 t}{6 + 3 \tan t} dt = \int \frac{du}{u} = \ln |u| + C = \boxed{\ln |6 + 3 \tan t| + C}$$

p.94, pr.10

(c) (10 Points) $\frac{d}{dx} (x^3 \log_{10} x) =$ **Solution:** By the product rule of derivatives, we have

$$\begin{aligned} \frac{d}{dx} (x^3 \log_{10} x) &= x^3 \frac{d}{dx} (\log_{10} x) + (\log_{10} x) \frac{d}{dx} (x^3) \\ &= x^3 \frac{1}{x \ln 10} + (\log_{10} x) (3x^2) \\ &= \frac{x^2}{\ln 10} + 3x^2 \log_{10} x \end{aligned}$$

p.94, pr.34

2. (a) (12 Points) Find the limit $\lim_{x \rightarrow 0^+} \left(\frac{3x+1}{x} - \frac{1}{\sin x} \right)$.

Solution:

$$\begin{aligned}
 \lim_{x \rightarrow 0^+} \left(\frac{3x+1}{x} - \frac{1}{\sin x} \right) &= \lim_{x \rightarrow 0^+} \left(\frac{(3x+1)\sin x - x}{x \sin x} \right) \\
 &= \lim_{x \rightarrow 0^+} \left(\frac{3 \sin x + (3x+1) \cos x - 1}{\sin x + x \cos x} \right) \\
 &= \lim_{x \rightarrow 0^+} \left(\frac{3 \cos x + 3 \cos x - (3x+1) \sin x - 0}{\cos x + \cos x - x \sin x} \right) \\
 &= \frac{3 \cos(0) + 3 \cos(0) - (3(0)+1) \sin(0) - 0}{\cos(0) + \cos(0) - (0) \sin(0)} \\
 &= \frac{3+3-(1)0-0}{1+1-0} = \frac{6}{2} = \boxed{3}
 \end{aligned}$$

p.72, pr.8

You may use L'Hôpital's rule wherever applicable.

- (b) (8 Points) Find the derivative $\frac{dy}{dv}$ of $y = \ln(\sinh v) - \frac{1}{2} \coth^2 v$ with respect to v .

Solution: We have

$$\begin{aligned}
 \frac{dy}{dv} &= \frac{d}{dv} \left(\ln(\sinh v) - \frac{1}{2} \coth^2 v \right) \\
 &= \frac{d}{dv} \ln(\sinh v) - \frac{1}{2} \frac{d}{dv} (\coth^2 v) \\
 &= \frac{1}{\sinh v} \frac{d}{dv} (\sinh v) - \frac{1}{2} (2 \coth v) \frac{d}{dv} (\coth v) \\
 &= \frac{\cosh v}{\sinh v} - \coth v (-\operatorname{csch}^2 v) = \coth v + \coth v \operatorname{csch}^2 v \\
 &= \boxed{\coth v (1 + \operatorname{csch}^2 v)}
 \end{aligned}$$

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p.82, pr.35

3. (a) (12 Points) Calculate $\sin^{-1}\left(\sin \frac{3\pi}{2}\right) + \sec^{-1}(2)$.

Solution: Note that it would be wrong to give $3\pi/2$ as the answer, since $\sin^{-1}$ is always in the interval $[-\pi/2, \pi/2]$. Work the problem in steps as follows.

$$\sin^{-1}\left(\sin \frac{3\pi}{2}\right) = \sin^{-1}(-1) = \boxed{-\pi/2}.$$

Moreover

$$\sec^{-1}(2) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}.$$

and so

$$\sin^{-1}\left(\sin \frac{3\pi}{2}\right) + \sec^{-1}(2) = -\frac{\pi}{2} + \frac{\pi}{3} = \boxed{-\frac{\pi}{6}}$$

p.72, pr.8

First find them separately and then add them up.

- (b) (13 Points) Evaluate the integral $\int \sec^4(4\theta) \, d\theta$.

Solution: First notice that

$$\begin{aligned} \int \sec^4(4\theta) \, d\theta &= \int \sec^2(4\theta) \sec^2(4\theta) \, d\theta \\ &= \int (1 + \tan^2(4\theta)) \sec^2(4\theta) \, d\theta \end{aligned}$$

Let $u = \tan(4\theta)$ and so $du = 4\sec^2(4\theta) \, d\theta$. Hence

$$\begin{aligned} \int \sec^4(4\theta) \, d\theta &= \int \sec^2(4\theta) \sec^2(4\theta) \, d\theta \\ &= \frac{1}{4} \int \underbrace{(1 + \tan^2(4\theta))}_{1+u^2} \underbrace{\sec^2(4\theta) \, d\theta}_{du} \\ &= \frac{1}{4} \int (1 + u^2) \, du \\ &= \frac{1}{4} \left[u + \frac{1}{3} u^3 \right] + C \\ &= \boxed{\frac{1}{4} \tan(4\theta) + \frac{1}{12} \tan^3(4\theta) + C} \end{aligned}$$

p.83, pr.52

4. (a) (13 Points) Evaluate the integral $\int x(\ln x)^2 dx$.

Solution: We shall integrate by parts. Let $u = (\ln x)^2$ and so $dv = dx$. Then $du = 2(\ln x) \frac{1}{x} dx$ and choose $v = x$. Therefore

$$\begin{aligned}\int x(\ln x)^2 dx &= \int u dv = uv - \int v du \\ &= (\ln x)^2(x) - \int x \cdot 2(\ln x) \frac{1}{x} dx \\ &= x(\ln x)^2 - 2 \int \ln x dx \\ &= x(\ln x)^2 - 2(x \ln x - x) + C \\ &= \boxed{x(\ln x)^2 - 2x \ln x + 2x + C}\end{aligned}$$

p.95, pr.68

- (b) (12 Points) Evaluate the integral $\int \frac{2x^3 - 2x^2 + 1}{x^2 - x} dx$.

Solution: By long division of polynomials, we have

$$\begin{array}{r} 2x+0 \\ x^2-x+0 \overline{) 2x^3-2x^2+0x+1} \\ \underline{-2x^3+2x^2+0x} \\ 0x^2+0x+1 \\ \underline{0x^2+0x+0} \\ 0x+1 \end{array}$$

Therefore,

$$\frac{2x^3 - 2x^2 + 0x + 1}{x^2 - x} = 2x + \frac{1}{x^2 - x}.$$

We decompose the integrand in the following way:

$$\frac{1}{x^2 - x} = \frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

Clearing the fractions changes to

$$1 = A(x-1) + Bx \Rightarrow 1 = x(A+B) - A \Rightarrow -A = 1, A+B = 0 \Rightarrow A = -1, B = 1.$$

Thus,

$$\begin{aligned}\int \frac{2x^3 - 2x^2 + 1}{x^2 - x} dx &= \int \left(2x - \frac{1}{x} + \frac{1}{x-1} \right) dx = \int 2x dx - \int \frac{1}{x} dx + \int \frac{1}{x-1} dx \\ &= \boxed{x^2 - \ln|x| + \ln|x-1| + C}\end{aligned}$$

p.112, pr.26