Math 113 Summer 2016 **First Exam** July 12, 2016

Your Name / Ad - Soyad (**75 min.**) Signature / İmza

Student ID # / Öğrenci No (**use a blue pen!**)

Problem	1	2	3	4	Total
Points:	30	20	25	25	100
Score:					

1. Find the following limits.

(a) (15 Points) $\displaystyle\lim_{h\to 0^+} \frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}$.

Solution: We have

$$\begin{aligned}\lim_{h\to 0^+} \frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h} &= \lim_{h\to 0^+} \frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}\cdot\frac{\sqrt{h^2+4h+5}+\sqrt{5}}{\sqrt{h^2+4h+5}+\sqrt{5}}\\ &= \lim_{h\to 0^+} \frac{h^2+4h+\cancel{5}-\cancel{5}}{h\left(\sqrt{h^2+4h+5}+\sqrt{5}\right)}\\ &= \lim_{h\to 0^+} \frac{\cancel{h}(h+4)}{\cancel{h}\left(\sqrt{h^2+4h+5}+\sqrt{5}\right)}\\ &= \lim_{h\to 0^+} \frac{h+4}{\sqrt{h^2+4h+5}+\sqrt{5}}\\ &= \frac{0+4}{\sqrt{0^2+(4)(0)+5}+\sqrt{5}} == \frac{4}{\sqrt{5}+\sqrt{5}} = \frac{4}{2\sqrt{5}} = \boxed{\frac{2}{\sqrt{5}}}\end{aligned}$$

p.72, pr.15

You are not allowed to use L'Hôpital's rule.

(b) (7 Points) $\displaystyle\lim_{\theta\to -\infty} \frac{\cos\theta}{3\theta}$.

Solution: We know that

$$-1 \le \cos\theta \le 1, \quad \forall\theta \Rightarrow -\frac{1}{3\theta} \le \frac{\cos\theta}{3\theta} \le \frac{1}{3\theta}, \quad \forall\theta \ne 0$$

and also $\displaystyle\lim_{\theta\to -\infty} \frac{-1}{3\theta} = \lim_{\theta\to -\infty} \frac{1}{3\theta} = 0$. Hence by Sandwich Theorem, $\boxed{\displaystyle\lim_{\theta\to -\infty} \frac{\cos\theta}{3\theta} = 0}$.

p.94, pr.10

You are not allowed to use L'Hôpital's rule. May be Sandwich Theorem

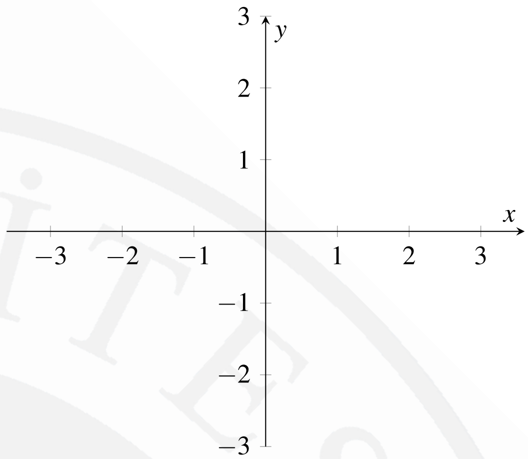
(c) (8 Points) $\displaystyle\lim_{x\to -\infty} \frac{\sqrt{x^2+1}}{x+1}$.

Solution: As $x \to -\infty$, we have $\sqrt{x^2} = |x| = -x$ and so $x = -\sqrt{x^2}$

$$\begin{aligned}\lim_{x\to -\infty} \frac{\sqrt{x^2+1}}{x+1} &= \lim_{x\to -\infty} \frac{\sqrt{x^2+1}}{x+1}\cdot\frac{1/x}{1/x}\\ &= \lim_{x\to -\infty} \frac{\dfrac{\sqrt{x^2+1}}{-\sqrt{x^2}}}{1+1/x}\\ &= \lim_{x\to -\infty} \frac{-\sqrt{1+1/x^2}}{1+1/x} = \boxed{-1}\end{aligned}$$

p.94, pr.34

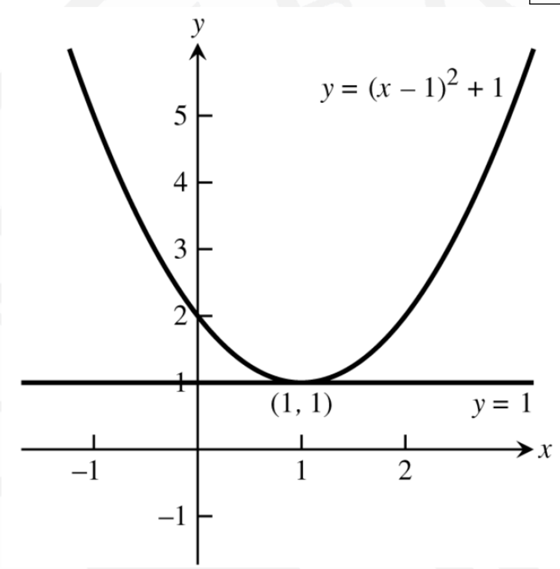
2. (a) (10 Points) Find an equation for the tangent to the curve $y = (x-1)^2 + 1$ at point $(1,1)$. Then sketch this curve and tangent together.



Solution: First the slope for the line at $(1,1)$ is

$$m = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{[(1+h-1)^2+1]-[(1-1)^2+1]}{h}$$
$$= \lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0}(h) = 0.$$

Then the equation for the tangent is $y-1 = 0(x-1) \Rightarrow \boxed{y=1}$.



p.72, pr.8

(b) (10 Points) Find the value(s) of c if the function

$$g(x) = \begin{cases} \dfrac{x^2-16}{x^2-3x-4} & x \neq 4 \\ c, & x = 4 \end{cases}$$

is continuous at $x = 4$.

Solution: First note that

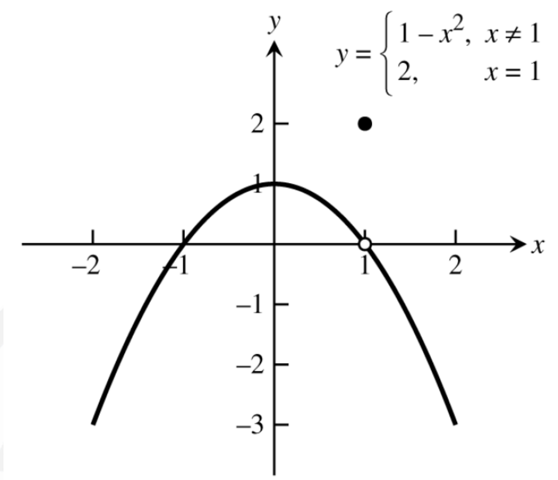
$$\lim_{x\to 4} g(x) = \lim_{x\to 4} \frac{x^2-16}{x^2-3x-4} = \lim_{x\to 4} \frac{(x-4)(x+4)}{(x-4)(x+1)} = \lim_{x\to 4} \frac{\cancel{(x-4)}(x+4)}{\cancel{(x-4)}(x+1)}$$
$$= \lim_{x\to 4} \frac{x+4}{x+1} = \frac{4+4}{4+1} = \frac{8}{5}$$

Now if $g(x)$ is continuous at $x = 4$, then $\lim_{x\to 4} g(x) = g(4)$, that is $\boxed{c = \frac{8}{5}}$.

p.83, pr.40

3. (a) (12 Points) Graph $f(x) = \begin{cases} 1-x^2 & x \neq 1 \\ 2, & x = 1 \end{cases}$ and find $\lim_{x\to 1} f(x)$.

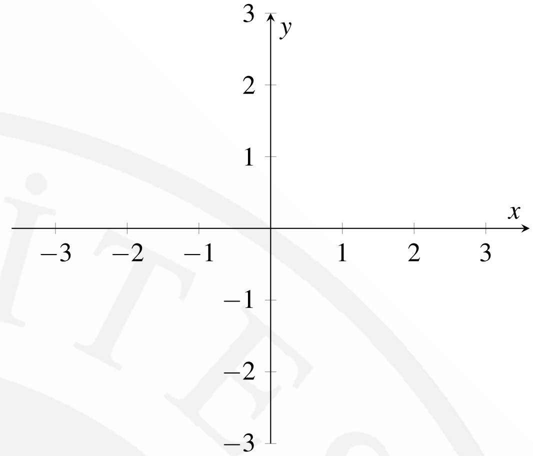
Solution: Here is the required graph.



The definition of limit ignores the behavior at $x = 1$. For $\lim_{x\to 1} f(x)$, we consider the y-values as x gets closer to 1 with $x \neq 1$. That is, we follow the graph toward $x = 1$ from the left ($x < 1$) and right ($x > 1$).

Hence $\lim_{x\to 1} f(x) = \lim_{x\to 1}(1-x^2) = 1 - 1^2 = 1 - 1 = \boxed{0}$

p.72, pr.8



(b) (13 Points) Explain why the equation $\cos x = x$ has *at least one solution*.

Solution: Solutions to the equation $\cos(x) = x$ are obviously the same thing as solutions to the equation $\cos(x) - x = 0$. These, in turn, are the same thing as zeros of the function $f(x) = \cos(x) - x$. This function is *continuous* for all x, and one has that

$f(0) = \cos(0) - 0 = 1 - 0 = 1$

is positive, whereas

$f(\pi/2) = \cos(\pi/2) - \pi/2 = 0 - \pi/2 = -\pi/2$

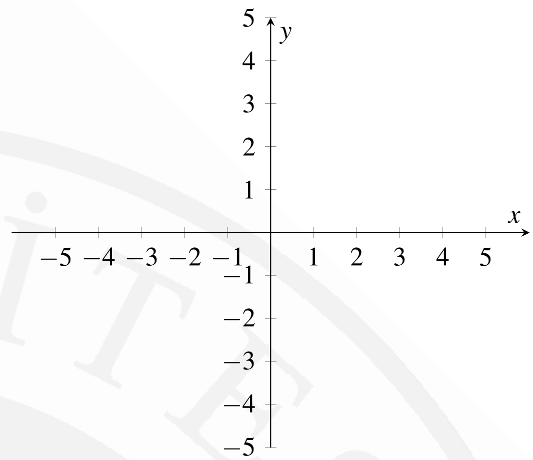
is negative.

By the intermediate value theorem, there has to be some c between 0 and $\pi/2$ for which $f(c) = 0$. [And this umber c is of course a solution to the original equation $\cos(x) = x$]

p.83, pr.52

You may use Intermediate Value Theorem.

4. (a) (13 Points) Find all the asymptotes and graph $y = \dfrac{2x}{x+1}$.



Solution: First notice that

$$y = \frac{2x}{x+1} = 2\left(\frac{x+1-1}{x+1}\right) = 2\left(1 - \frac{1}{x+1}\right).$$

Then

$$\lim_{x\to\pm\infty} \frac{2x}{x+1} = \lim_{x\to\pm\infty} 2\left(1-\frac{1}{x+1}\right) = 2(1-0) = 2$$

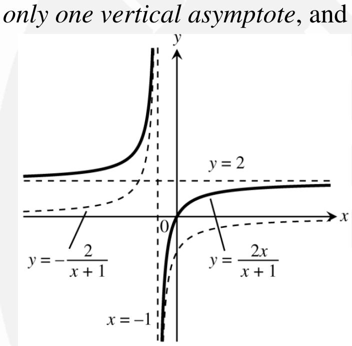
and so $\boxed{y=2}$ *is the (only) horizontal asymptote*. Moreover, since

$$\lim_{x\to -1^+} \frac{2x}{x+1} = \lim_{x\to -1^+} 2\left(1-\frac{1}{x+1}\right) = 2(1-\infty) = -\infty$$

and

$$\lim_{x\to -1^-} \frac{2x}{x+1} = \lim_{x\to -1^-} 2\left(1-\frac{1}{x+1}\right) = 2(1+\infty) = +\infty,$$

the graph has *only one vertical asymptote*, and it is $\boxed{\text{x=-1}}$.



p.95, pr.68

(b) (12 Points) Use the formula $f'(x) = \lim_{z\to x} \dfrac{f(z)-f(x)}{z-x}$ to find the derivative of $g(x) = 1+\sqrt{x}$.

Solution: Here $f(z) = g(z) = 1+\sqrt{z}$ and $f(x) = g(x) = 1+\sqrt{x}$ and so

$$\begin{aligned} f'(x) &= \lim_{z\to x} \frac{f(z)-f(x)}{z-x} = \lim_{z\to x} \frac{g(z)-g(x)}{z-x} \\ &= \lim_{z\to x} \frac{\cancel{1}+\sqrt{z}-(\cancel{1}+\sqrt{x})}{z-x} \\ &= \lim_{z\to x} \frac{\cancel{\sqrt{z}-\sqrt{x}}}{\cancel{(\sqrt{z}-\sqrt{x})}(\sqrt{z}+\sqrt{x})} \\ &= \lim_{z\to x} \frac{1}{(\sqrt{z}+\sqrt{x})} = \frac{1}{(\sqrt{x}+\sqrt{x})} = \boxed{\frac{1}{2\sqrt{x}}} \end{aligned}$$

p.112, pr.26