Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

Problem	Points	Score
1	20	
2	30	
3	25	
4	25	
Total:	100	

Do not write in the table to the right.

1. (a) 10 Points $\lim_{x\to 0}\left(\frac{1}{x^4}-\frac{1}{x^2}\right)=?$

Solution: We have

$$\lim_{x\to 0}\left(\frac{1}{x^4}-\frac{1}{x^2}\right)=\lim_{x\to 0}\frac{1-x^2}{x^4}=+\infty.$$

p.431, pr.94

(b) 10 Points $\int_1^4 \frac{\cosh(\sqrt{x})}{\sqrt{x}}\,dx=?$

Solution: Substitute $u=\sqrt{x}$ and so $du=\frac{1}{2\sqrt{x}}\,dx$. When $x=1$, we have $u=1$ and when $x=4$, we have $u=2$. Hence

$$\int_1^4 \frac{\cosh(\sqrt{x})}{\sqrt{x}}\,dx = 2\int_1^4 \frac{\cosh(\sqrt{x})}{2\sqrt{x}}\,dx = 2\int_1^2 \cosh u\,du = (2\sinh u)\big|_1^2 = 2(\sinh 2-\sinh 1) = 2\left(\frac{e^2-e^{-2}}{2}-\frac{e^1-e^{-1}}{2}\right)$$

p.448, pr.22

2. (a) 15 Points $\displaystyle\int_0^{\sqrt{3}/2} \frac{4x^2}{(1-x^2)^{3/2}}\,\mathrm{d}x = ?$

Solution: We use the method of trigonometric substitution. Let $x = \sin\theta$ and so $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$. When $x = 0$, we have $\theta = 0$. When $x = \frac{\sqrt{3}}{2}$, we have $\theta = \frac{\pi}{3}$. Now $1 - x^2 = 1 - \sin^2\theta = \cos^2\theta$ and so $(1-x^2)^{3/2} = (\cos^2\theta)^{3/2} = \cos^3\theta$. This yields

$$\begin{aligned}
\int_0^{\sqrt{3}/2} \frac{4x^2}{(1-x^2)^{3/2}}\,\mathrm{d}x &= \int_0^{\pi/3} \frac{4\sin^2\theta}{\cos^3\theta}\cos\theta\,\mathrm{d}\theta = 4\int_0^{\pi/3}\tan^2\theta\,\mathrm{d}\theta \\
&= 4\int_0^{\pi/3}(\sec^2\theta - 1)\,\mathrm{d}\theta \\
&= 4\,(\tan\theta - \theta)\big|_0^{\pi/3} = 4(\tan(\pi/3) - \pi/3) \\
&= \boxed{4(\sqrt{3} - \pi/3)}
\end{aligned}$$

p.452, pr.23

(b) 15 Points Evaluate the integral $\displaystyle\int_2^{\infty} \frac{2\,\mathrm{d}t}{t^2-1}$.

Solution: First, by partial fraction decomposition, notice that

$$\frac{2}{t^2-1} = \frac{A}{t-1} + \frac{B}{t+1} \Rightarrow A(t+1) + B(t-1) = 2.$$

When $t = 1$, we have $2A = 2$ and so $A = 1$. Similarly, for $t = -1$, we have $-2B = 2$ yielding $B = -1$. Hence

$$\int \frac{2\,\mathrm{d}t}{t^2-1} = \int \left(\frac{1}{t-1} + \frac{-1}{t+1}\right)\mathrm{d}t = \ln|t-1| - \ln|t+1| + C = \ln\left|\frac{t-1}{t+1}\right| + C$$

Therefore,

$$\int_2^{\infty} \frac{2\,\mathrm{d}t}{t^2-1} = \lim_{b\to\infty}\int_2^{b} \frac{2\,\mathrm{d}t}{t^2-1} = \lim_{b\to\infty}\left[\ln\left|\frac{t-1}{t+1}\right|\right]_2^b = \lim_{b\to\infty}\left(\ln\frac{b-1}{b+1} - \ln\frac{2-1}{2+1}\right) = \ln 1 - \ln\frac{1}{3} = \boxed{\ln 3}$$

p.487, pr.12

3. (a) [13 Points] Find a plane through $P_0(2,1,-1)$ and perpendicular to the line of intersection of planes $2x+y-z=3$ and $x+2y+z=2$.

Solution: A vector normal the first given plane is $\mathbf{n_1} = 2\mathbf{i}+\mathbf{j}-\mathbf{k}$ and a vector normal to the second plane is $\mathbf{n_2} = \mathbf{i}+2\mathbf{j}+\mathbf{k}$. We first find a vector $\mathbf{n}$ that is normal to the plane we want to determine. Then $\mathbf{n}$ must be parallel to the vector

$$\begin{aligned} \mathbf{n_1}\times\mathbf{n_2} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}\mathbf{k} \\ &= 3\mathbf{i}-3\mathbf{j}+3\mathbf{k} \end{aligned}$$

we may take $\mathbf{n} = \mathbf{i}-\mathbf{j}+\mathbf{k}$. Therefore, the equation of the desired plane is

$$\mathbf{n} = \mathbf{n}\cdot\vec{P_0P} = 0 \Rightarrow (\mathbf{i}-\mathbf{j}+\mathbf{k})\cdot((x-2)\mathbf{i}+(y-1)\mathbf{j}+(z+1)\mathbf{k}) = 0 \Rightarrow (x-2)-(y-1)+(z+1) = 0 \Rightarrow \boxed{x-y+z=0.}$$

p.695, pr.31

(b) [12 Points] If $z^3-xy+yz+y^3-2=0$, find $\left.\frac{\partial z}{\partial x}\right|_{(1,1,1)}$ and $\left.\frac{\partial z}{\partial y}\right|_{(1,1,1)}$.

Solution: We can do this in two ways. First we can differentiate implicitly with respect to x (remember y is held constant).

$$\begin{aligned} \frac{\partial}{\partial x}\left(z^3-xy+yz+y^3-2\right) &= \frac{\partial}{\partial x}(0) \\ 3z^2\frac{\partial z}{\partial x}-y+y\frac{\partial z}{\partial x} &= 0 \\ (3z^2+y)\frac{\partial z}{\partial x} &= y \\ \frac{\partial z}{\partial x} &= \frac{y}{3z^2+y} \\ \left.\frac{\partial z}{\partial x}\right|_{(1,1,1)} &= \boxed{\frac{1}{4}} \end{aligned}$$

Similarly, differentiating with respect to y gives (by treating x constant), we have

$$\begin{aligned} \frac{\partial}{\partial y}\left(z^3-xy+yz+y^3-2\right) &= \frac{\partial}{\partial y}(0) \\ 3z^2\frac{\partial z}{\partial y}-x+z+y\frac{\partial z}{\partial y}+3y^2 &= 0 \\ (3z^2+y)\frac{\partial z}{\partial y} &= x-z-3y^2 \\ \frac{\partial z}{\partial y} &= \frac{x-z-3y^2}{3z^2+y} \\ \left.\frac{\partial z}{\partial y}\right|_{(1,1,1)} &= \boxed{-\frac{3}{4}} \end{aligned}$$

As an alternative method, we differentiate implicitly the given equation with respect to first x and then with respect to y. Let $F(x,y,z) = z^3-xy+yz+y^3-2=0$. Then $F_x(x,y,z) = -y$, $F_y(x,y,z) = -x+z+3y^2$ and $F_z(x,y,z) = 3z^2+y$. Therefore, by implicit differentiation formulas (Theorem 8, page 780 of the textbook)

$$\begin{aligned} \left.\frac{\partial z}{\partial x}\right|_{(1,1,1)} &= -\left.\frac{F_x}{F_z}\right|_{(1,1,1)} = -\left.\frac{-y}{3z^2+y}\right|_{(1,1,1)} = \boxed{\frac{1}{4}} \\ \left.\frac{\partial z}{\partial y}\right|_{(1,1,1)} &= -\left.\frac{F_y}{F_z}\right|_{(1,1,1)} = -\left.\frac{-x+z+3y^2}{3z^2+y}\right|_{(1,1,1)} = \boxed{-\frac{3}{4}}. \end{aligned}$$

p.452, pr.24

4. (a) 10 Points Determine if $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$ converges or diverges. Give reason.

Solution: For each $n = 1, 2, 3, \cdots$, let $a_n = \frac{n!}{(2n+1)!} > 0$. Use the Ratio Test. Then, for each $n = 1, 2, 3, \cdots$, we have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{n!} = \frac{(n+1)\cancel{n!}}{(2n+3)(2n+2)\cancel{(2n+1)!}} \frac{\cancel{(2n+1)!}}{\cancel{n!}} = \frac{(n+1)}{(2n+3)(2n+2)}.$$

Hence $\rho := \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)}{(2n+3)(2n+2)} = \lim_{n\to\infty} \frac{\cancel{n^2}(\frac{1}{n} + \frac{1}{n^2})}{\cancel{n^2}(2+\frac{3}{n})(2+\frac{2}{n})} = \frac{0}{4} = \boxed{0} < 1$. Therefore, the series *converges* by the Ratio Test.

p.695, pr.29

(b) 15 Points Find the radius and interval of convergence for $\sum_{n=0}^{\infty} \frac{(n+1)(2x+1)^n}{(2n+1)2^n}$.

Solution: The center of convergence is $c = -\frac{1}{2}$Let $u_n = \frac{(n+1)(2x+1)^n}{(2n+1)2^n}$ where $n = 0, 1, 2, \cdots$. Then

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+2)(2x+1)^{n+1}}{(2n+3)2^{n+1}} \cdot \frac{(2n+1)2^n}{(n+1)(2x+1)^n}\right| = \left|\frac{(n+2)(2n+1)\cancel{2^n}}{(2n+3)(n+1)\cancel{2^n}\cdot 2} \frac{\cancel{(2x+1)^n}(2x+1)}{\cancel{(2x+1)^n}}\right| = |2x+1| \frac{(n+2)(2n+1)}{2(2n+3)(n+1)}$$

Now

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = |2x+1| \lim_{n\to\infty} \frac{\cancel{n^2}(1+\frac{2}{n}2)(2+\frac{1}{n})}{2\cancel{n^2}(2+\frac{3}{n})(1+\frac{1}{n})} = \frac{|2x+1|}{2}$$

Then, according to absolute ratio test the power series converges absolutely if $|2x+1| < 2$, diverges if $|2x+1| > 2$ and is inconclusive if $|2x+1| = 2$. That is, the series converges absolutely if $-2 < 2x+1 < 2$ equivalently if $-\frac{3}{2} < x < \frac{1}{2}$ and diverges if $x < -\frac{3}{2}$ or if $x > \frac{1}{2}$. Now we must test the series for the endpoints.

For $x = -\frac{3}{2}$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(n+1)(2(-\frac{3}{2})+1)^n}{(2n+1)2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{(2n+1)}$$

which *diverges* by the nth Term Test.

For $x = \frac{1}{2}$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(n+1)(2(\frac{1}{2})+1)^n}{(2n+1)2^n} = \sum_{n=0}^{\infty} \frac{(n+1)}{(2n+1)}$$

which also *diverges* by the nth Term Test. Now we conclude that the interval of convergence is $-\frac{3}{2} < x < \frac{1}{2}$ and radius of convergence is $R = 1$

p.385, pr.88