Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place [a box around your answer] to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 75 min.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. Given: The surface with equation $x^3+y^3+z^3=5xyz$ and the point $P_0(2,1,1)$ on it.

 (a) [13 Points] Find $\left.\frac{\partial z}{\partial x}\right|_{(2,1,1)}$ and $\left.\frac{\partial z}{\partial y}\right|_{(2,1,1)}$.

Solution: We can do this in two ways.

First we can differentiate implicitly with respect to x (remember y is held constant).

$$\frac{\partial}{\partial x}\left(x^3+y^3+z^3\right)=\frac{\partial}{\partial x}(5xyz)$$
$$3x^2+3z^2\frac{\partial z}{\partial x}=5yz+5xy\frac{\partial z}{\partial x}$$
$$(3z^2-5xy)\frac{\partial z}{\partial x}=5yz-3x^2$$
$$\frac{\partial z}{\partial x}=\frac{5yz-3x^2}{3z^2-5xy}$$
$$\Rightarrow \left.\frac{\partial z}{\partial x}\right|_{(2,1,1)}$$
$$=\frac{5(1)(1)-3(2)^2}{3(1)^2-5(2)(1)}=\boxed{1}.$$

Similarly, differentiating with respect to y (by treating x constant), we have

$$\frac{\partial}{\partial y}\left(x^3+y^3+z^3\right)=\frac{\partial}{\partial y}(5xyz)$$
$$3y^2+3z^2\frac{\partial z}{\partial y}-x+z+y\frac{\partial z}{\partial y}+3y^2=5xz+5xy\frac{\partial z}{\partial x}$$
$$(3z^2-5xy)\frac{\partial z}{\partial y}=5xz-3y^2$$
$$\frac{\partial z}{\partial y}=\frac{5xz-3y^2}{3z^2-5xy}$$
$$\Rightarrow \left.\frac{\partial z}{\partial y}\right|_{(2,1,1)}$$
$$=\frac{5(2)(1)-3(1)^2}{3(1)^2-5(2)(1)}=\boxed{-1}.$$

As an alternative method, let $F(x,y,z)=x^3+y^3+z^3-5xyz=0$. Then $F_x(x,y,z)=3x^2-5yz$, $F_y(x,y,z)=3y^2-5yz$ and

$F_z(x,y,z) = 3z^2 - 5xz$. Therefore, by implicit differentiation formulas (Theorem 8, page 780 of the textbook)

$$\left.\frac{\partial z}{\partial x}\right|_{(2,1,1)} = -\left.\frac{F_x}{F_z}\right|_{(2,1,1)} = -\left.\frac{3x^2-5yz}{3z^2-5xz}\right|_{(2,1,1)} = \boxed{\frac{7}{7}=1}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(2,1,1)} = -\left.\frac{F_y}{F_z}\right|_{(2,1,1)} = -\left.\frac{3y^2-5yz}{3z^2-5xz}\right|_{(2,1,1)} = \boxed{-\frac{-7}{-7}=-1}.$$

p.583, pr.32

(b) 12 Points Find the equation for plane tangent to this surface at P_0.

Solution: Since

$$F_x(x,y,z) = 3x^2 - 5yz \Rightarrow F_x(2,1,1) = 3(2)^2 - 5(1)(1) = 7$$
$$F_y(x,y,z) = 3y^2 - 5yz \Rightarrow F_y(2,1,1) = 3(1)^2 - 5(2)(1)0 = -7$$
$$F_z(x,y,z) = 3z^2 - 5xy \Rightarrow F_z(2,1,1) = 3(1)^2 - 5(2)(1) = -7,$$

we have $\nabla F(2,1,1) = 7\mathbf{i} - 7\mathbf{j} - 7\mathbf{k}$.
Hence an equation of the plane tangent to the surface at P_0 is

$$\nabla F(2,1,1)\cdot((x-2)\mathbf{i} + (y-1)\mathbf{j} + (z-1)\mathbf{k}) = 0 \Rightarrow z-1 = (x-2)-(y-1);$$

that is, $\boxed{x-y-z=0}$.

p.452, pr.24

2. (a) 13 Points $\displaystyle\int_1^4 \frac{3\sinh\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x = ?$

Solution: Substitute $u = \sqrt{x}$ and so $du = \dfrac{1}{2\sqrt{x}}\,\mathrm{d}x$. When $x = 1$, we have $u = 1$ and when $x = 4$, we have $u = 2$. Hence

$$\int_1^4 \frac{3\sinh(\sqrt{x})}{\sqrt{x}}\,\mathrm{d}x = 6\int_1^4 \frac{\sinh(\sqrt{x})}{2\sqrt{x}}\,\mathrm{d}x = 6\int_1^2 \sinh u\,\mathrm{d}u = 6\,(\cosh u)|_1^2 = 6(\cosh 2 - \cosh 1) = 6\left(\frac{\mathrm{e}^2+\mathrm{e}^{-2}}{2} - \frac{\mathrm{e}^1+\mathrm{e}^{-1}}{2}\right)$$

p.448, pr.22 p.487, pr.6

(b) 12 Points $\displaystyle\int \frac{\mathrm{d}x}{1+(3x+1)^2} = ?$

Solution: Let $u = 3x+1$. Then $\mathrm{d}u = 3\,\mathrm{d}x$. Thus, we have

$$\int \frac{\mathrm{d}x}{1+(3x+1)^2} = \frac{1}{3}\int \frac{3\,\mathrm{d}x}{1+(3x+1)^2} = \frac{1}{3}\int \frac{\mathrm{d}u}{1+u^2} = \frac{1}{3}\tan^{-1}u + C = \frac{1}{3}\tan^{-1}(3x+1) + C$$

p.413, pr.58

3. (a) 14 Points Does the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\sqrt{n}}$ converge absolutely, conditionally, or diverge? Justify your answer.

Solution: This is an alternating series of the form $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$ with $a_n=\dfrac{1}{1+\sqrt{n}}>0$ for all $n\geq 1$. Using the Alternating Series Test (AST),

- $\displaystyle\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{1}{1+\sqrt{n}}=0$

and

- $\displaystyle\frac{a_{n+1}}{a_n}=\frac{1}{1+\sqrt{n+1}}\cdot\frac{1+\sqrt{n}}{1}=\frac{1+\sqrt{n}}{1+\sqrt{n+1}}<1$ for all $n\geq 1$,

so $a_{n+1}<a_n$ for all $n\geq 1$, so the series converges.
But by the Limit Comparison Test (LCT), letting

$$a_n=\frac{1}{1+\sqrt{n}},\quad b_n=\frac{1}{\sqrt{n}}$$

we have

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\sqrt{n}}{1+\sqrt{n}}\cdot\frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}}=\lim_{n\to\infty}\frac{1}{\frac{1}{\sqrt{n}}+1}=1$$

so $0<c=1<\infty$ and $\sum\frac{1}{\sqrt{n}}$ diverges implies $\sum\frac{1}{1+\sqrt{n}}$ diverges too.

Therefore $\displaystyle\sum_{n=1}^{\infty}|(-1)^n a_n|=\sum_{n=1}^{\infty}\frac{1}{1+\sqrt{n}}$ diverges. So $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\sqrt{n}}$ *converges conditionally*.

p.573, pr.18

(b) 11 Points Evaluate the integral $\displaystyle\int(x+1)^2e^x\,dx$

Solution: We shall integrate by parts twice. Let $u=(x+1)^2$ and $dv=e^x\,dx$. Then $du=2(x+1)\,dx$ and choose $v=e^x$. Therefore

$$\begin{aligned}\int(x+1)^2e^x\,dx&=\int u\,dv=uv-\int v\,du\\&=(x+1)^2e^x-\int\underbrace{2(x+1)}_{u}\underbrace{e^x\,dx}_{dv}\\&=(x+1)^2e^x-\left[\underbrace{2(x+1)}_{u}\underbrace{e^x}_{v}-\int\underbrace{e^x}_{v}\underbrace{2\,dx}_{du}\right]=(x+1)^2e^x-2(x+1)e^x+2\int e^x\,dx\\&=\boxed{(x+1)^2e^x-2(x+1)e^x+2e^x+C}\end{aligned}$$

p.489, pr.5

4. (a) 12 Points Find the distance from the point $Q(0,2,3)$ to the line $\mathscr{L}:\begin{cases} x = 3+2t, \\ y = 1+t, \\ z = -1+2t \end{cases}$

Solution: We shall use the distance formula $d = \dfrac{|\vec{PQ}\times\mathbf{v}|}{|\mathbf{v}|}$. Here (by taking $t = 0$) $P(3,1,-1)$ is a point on $\mathscr{L}$ and $\mathbf{v} = 2\mathbf{i}+\mathbf{j}+2\mathbf{k}$ is a vector that is parallel to $\mathscr{L}$. Now we have $\vec{PQ} = (0-3)\mathbf{i}+(2-1)\mathbf{j}+(3-(-1))\mathbf{k} = -3\mathbf{i}+\mathbf{j}+4\mathbf{k}$ and so

$$\begin{aligned} \vec{PQ}\times\mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & 4 \\ 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & 4 \\ 2 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 2 \\ 1 & 1 \end{vmatrix}\mathbf{k} \\ &= -2\mathbf{i}+14\mathbf{j}-5\mathbf{k} \end{aligned}$$

Therefore, we have

$$d = \frac{|\vec{PQ}\times\mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{4+196+25}}{\sqrt{4+1+4}} = \frac{\sqrt{225}}{\sqrt{9}} = \frac{15}{3} = \boxed{5}$$

p.695, pr.37

(b) 13 Points Find parametric equations for the line in which the planes $x+2y+z = 3$ and $x-4y+3z = 5$ intersect.

Solution: We begin by finding two points on the line. Any two on the line would do, but we choose to find the points where line pierces yz-plane and the xz-plane. We get the former by setting $x = 0$ and solving the resulting equations $\begin{cases} 2y+z = 3 \\ -4y+3z = 5 \end{cases}$ simultaneously. This yields the point $(0,2/5,11/5)$. Similarly, by setting $y = 0$, we get the equations $\begin{cases} x+z = 3 \\ x+3z = 5 \end{cases}$. Solving these yields the point $(2,0,1)$. Consequently a vector parallel to the required line is

$$\mathbf{v} = (2-0)\mathbf{i}+(0-(2/5))\mathbf{j}+(1-(11/5))\mathbf{k} = 2\mathbf{i}-\frac{2}{5}\mathbf{j}-\frac{6}{5}\mathbf{k}$$

We can clear the denominators out by multiplying this vector by 5, we can take $\mathbf{v}$ to be $\mathbf{v} = 10\mathbf{i}-2\mathbf{j}-6\mathbf{k}$. Using $(2,0,1)$ for (x_0,y_0,z_0), we get

$$\boxed{\mathscr{L}:\begin{cases} x = 2+10t, \\ y = 0-2t, \\ z = 1-6t \end{cases}}$$

An alternative solution is based on the fact that line of intersection for planes is perpendicular to both of their normals. The vector $\mathbf{n_1} := \mathbf{i}+2\mathbf{j}+\mathbf{k}$ is normal to the first plane; $\mathbf{n_2} := \mathbf{i}-4\mathbf{j}+3\mathbf{k}$ is normal to the second. Since

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & -4 & 3 \end{vmatrix} = 10\mathbf{i}-2\mathbf{j}-6\mathbf{k},$$

the vector $\mathbf{v} = 14\mathbf{i}+2\mathbf{j}+15\mathbf{k}$ is parallel to the required line. Next, find any point on the line of intersection, for example, $(2,0,1)$, and proceed as in the earlier solution. $\boxed{\mathscr{L}:\begin{cases} x = 2+10t, \\ y = -2t, \\ z = 1-6t \end{cases}}$

p.452, pr.24