Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- Calculators, cell phones off and away!.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

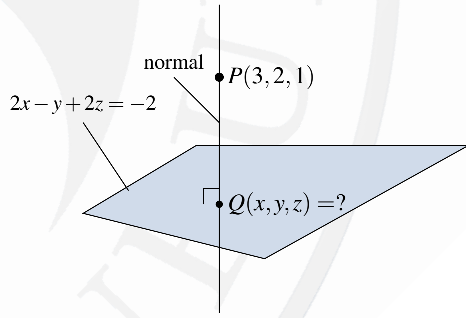
Do not write in the table to the right.

Problem	Points	Score
1	25	
2	22	
3	23	
4	30	
Total:	100	

1. (a) 12 Points Find the point in which the line through $P(3,2,1)$ normal to the plane $2x-y+2z=-2$ meets the plane.

Solution: Denote by $\mathscr{L}$ the line through $P(3,2,1)$ normal to the plane $2x-y+2z=-2$. Then write the parametric equations for $\mathscr{L}$.

$$\mathscr{L}:\begin{cases} x=3+2t \\ y=2-t \\ z=1+2t \end{cases}$$



So we are asked to find the point in which $\mathscr{L}$ meets $2x-y+2z=-2$. To this end, we simply substitute.

$$2(3+2t)-(2-t)+2(1+2t)=-2 \Rightarrow 6+6t-2+t+2+4t=-2 \Rightarrow 9t=-8$$

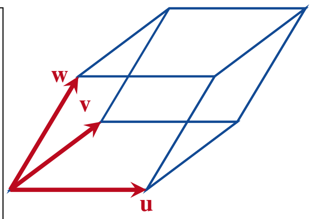
So $t=-8/9$. Therefore the point where by $\mathscr{L}$ meets the plane $2x-y+2z=-2$ has coordinates:

$$\mathscr{L}:\begin{cases} x=3+2(-8/9)=11/9 \\ y=2-(-8/9)=26/9 \\ z=1+2(-8/9)=-7/9 \end{cases}$$

That is, the point of intersection is $\boxed{(11/9,26/9/,-7/9)}$.

p.695, pr.37

(b) 13 Points Find the volume of the parallelepiped (box) determined by the vectors $\mathbf{u}=\mathbf{i}+\mathbf{j}-2\mathbf{k}$, $\mathbf{v}=-\mathbf{i}-\mathbf{k}$, $\mathbf{w}=2\mathbf{i}+4\mathbf{j}-2\mathbf{k}$ when they are placed with the same initial point.



Solution: If $\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $\mathbf{w} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, then

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (\mathbf{v}\times\mathbf{w})\cdot\mathbf{u} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (\mathbf{w}\times\mathbf{u})\cdot\mathbf{v}$$

which all have the same absolute value, since the interchanging two rows in a determinant does not change its absolute value, the volume of this parallelepiped is

$$|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}| = \begin{vmatrix} 1 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 4 & -2 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 4 & -2 \end{vmatrix} - \begin{vmatrix} -1 & -1 \\ 2 & -2 \end{vmatrix} + (-2)\begin{vmatrix} -1 & 0 \\ 2 & 4 \end{vmatrix}$$
$$= \boxed{8}$$

p.695, pr.37

2. (a) 10 Points *Evaluate* the integral $\int e^{\theta}\sin\theta\, d\theta$.

Solution: We integrate by parts with a twist. Let $u = e^{\theta}$ and so $dv = \sin\theta\ d\theta$. Then $du = e^{\theta}\ d\theta$ and choose $v = -\cos\theta$. Therefore

$$\int e^{\theta}\sin\theta\ d\theta = \int u\,dv = uv - \int v\,du \qquad (1)$$
$$= e^{\theta}(-\cos\theta) - \int(-\cos\theta)e^{\theta}\ d\theta \qquad (2)$$
$$= -e^{\theta}\cos\theta + \int e^{\theta}\cos\theta\ d\theta \qquad (3)$$

We next apply integration by parts to the integral on the right side of line (3). Letting

$$\int e^{\theta}\cos\theta\, d\theta = \begin{bmatrix} u = e^{\theta} & dv = \cos\theta\, d\theta \\ du = e^{\theta}\, d\theta & v = \sin\theta \end{bmatrix} = \underbrace{e^{\theta}}_{u}\underbrace{\sin\theta}_{v} - \int \underbrace{\sin\theta}_{v}\,\underbrace{e^{\theta}\, d\theta}_{du}$$

So we have

$$\int e^{\theta}\sin\theta\ d\theta = -e^{\theta}\cos\theta + e^{\theta}\sin\theta - \int e^{\theta}\sin\theta\ d\theta$$

Adding $\int e^{\theta}\sin\theta\ d\theta$ to both sides gives us

$$2\int e^{\theta}\sin\theta\ d\theta = e^{\theta}(\sin\theta + \cos\theta).$$

Finally, dividing both sides by 2 and adding a constant of integration, we have

$$\int e^{\theta}\sin\theta\ d\theta = \boxed{\frac{1}{2}e^{\theta}(\sin\theta + \cos\theta) + c}$$

Incidentally, this integral can be evaluated by using $dv = e^{\theta}\ d\theta$ for both the first and the second applications of the integration by parts formula.

p.652, pr.3

(b) 12 Points Investigate the convergence/divergence of the series $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$. Name the test you use.

○ Converges. ○ Diverges. Test Used: ____________

Solution: Use the Limit Comparison Test. Let $a_n = \frac{n}{n^2+1} > 0$ and choose $b_n = \frac{1}{n} > 0$.Then

$$\frac{a_n}{b_n} = \frac{n}{n^2+1}\frac{n}{1} = \frac{n^2}{n^2+1} = \frac{n^2}{n^2+1}\frac{1/n^2}{1/n^2} = \frac{1}{1+1/n^2} \to \frac{1}{1+0} = 1$$

So $0 < L = 1 < \infty$ and this series *diverges*, because $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

We can also use Direct Comparison Test. Notice that if $n \geq 1$, then $n^2+1 \leq n^+n^2 = 2n^2$ and so

$$\frac{1}{n^2+1} \geq \frac{1}{2n^2} > 0 \Rightarrow a_n = \frac{n}{n^2+1} \geq \frac{n}{2n^2} = \frac{1}{2n} = b_n > 0, \ \forall n \geq 1.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is harmonic series so it diverges. Therefore $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges too, as it is a scalar multiple of the harmonic series. Hence the given series is a larger series than the series of scalar multiple series. We conclude by Direct Comparison Test that the given series diverges.

A third way could be the Integral Test. Define $f(x) = \frac{1}{x^2+1}$ is positive, continuous and decreasing for all $x \geq 1$. So we compute the improper integral

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^b \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \left[\frac{1}{2}\ln(x^2+1)\right]_1^b = \lim_{b\to\infty} \left[\frac{1}{2}\ln(b^2+1) - \frac{1}{2}\ln 2\right] = \infty$$

Therefore the integral $\int_1^{\infty} \frac{x}{x^2+1}\,dx$ diverges. By the Integral Test, the series diverges too.

p.533, pr.95

3. (a) 13 Points Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(1,1,1)$ if $z^3 - xy + yz + y^3 - 2 = 0$.

Solution: We can do this in two ways.

First we can differentiate implicitly with respect to x (y is held constant).

$$\frac{\partial}{\partial x}\left(z^3 - xy + yz + y^3 - 2\right) = \frac{\partial}{\partial x}(0)$$

$$3z^2\frac{\partial z}{\partial x} - y + y\frac{\partial z}{\partial x} = 0$$

$$(3z^2+y)\frac{\partial z}{\partial x} = y$$

$$\frac{\partial z}{\partial x} = \frac{y}{3z^2+y}$$

$$\Rightarrow \left.\frac{\partial z}{\partial x}\right|_{(1,1,1)} = \frac{1}{3(1)^2+1} = \boxed{\frac{1}{4}}.$$

Similarly, differentiating with respect to y (treating x constant), we have

$$\frac{\partial}{\partial y}\left(z^3 - xy + yz + y^3 - 2\right) = \frac{\partial}{\partial y}(0)$$

$$3z^2\frac{\partial z}{\partial y} - x + z + y\frac{\partial z}{\partial y} + 3y^2 = 0$$

$$(3z^2+y)\frac{\partial z}{\partial y} = x - z - 3y^2$$

$$\frac{\partial z}{\partial y} = \frac{x-z-3y^2}{3z^2+y}$$

$$\Rightarrow \left.\frac{\partial z}{\partial y}\right|_{(1,1,1)} = \boxed{-\frac{3}{4}}.$$

As an alternative method, let $F(x,y,z) = z^3 - xy + yz + y^3 - 2 = 0$. Then $F_x(x,y,z) = -y$, $F_y(x,y,z) = -x + z + 3y^2$ and $F_z(x,y,z) = 3z^2 + y$. Therefore, by implicit differentiation formulas (Theorem 8, page 780 of the textbook)

$$\left.\frac{\partial z}{\partial x}\right|_{(1,1,1)} = -\left.\frac{F_x}{F_z}\right|_{(1,1,1)} = -\left.\frac{-y}{3z^2+y}\right|_{(1,1,1)} = \boxed{\frac{1}{4}}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,1,1)} = -\left.\frac{F_y}{F_z}\right|_{(1,1,1)} = -\left.\frac{-x+z+3y^2}{3z^2+y}\right|_{(1,1,1)} = \boxed{-\frac{3}{4}}.$$

p.879, pr.42

(b) 10 Points Let $z = \sqrt{y-x}$. Find the equation for tangent plane to this surface at the point $(1,2,1)$.

Solution: Let $F(x,y,z) = \sqrt{y-x} - z$. Then

$$F_x(x,y,z) = -\frac{1}{2}(y-x)^{-1/2}$$

$$F_y(x,y,z) = \frac{1}{2}(y-x)^{-1/2}$$

$$F_z(x,y,z) = -1$$

At $(1,2,1)$, we have

$$F_x(1,2,1) = -\frac{1}{2}(2-1)^{-1/2} = -\frac{1}{2}$$
$$F_y(1,2,1) = \frac{1}{2}(2-1)^{-1/2} = -\frac{1}{2}$$
$$F_z(1,2,1) = -1.$$

Hence the tangent plane at $(1,2,1)$ is

$$-\frac{1}{2}(x-1) + \frac{1}{2}(y-2) - (z-1) = 0 \Rightarrow \boxed{x - y + 2z = 1}$$

p.879, pr.34

4. (a) 15 Points Find two numbers a and b with $a \le b$ such that $\int_a^b \left(6 - x - x^2\right) dx$ has its largest value.

Solution: Let $F(a,b) = \int_a^b \left(6 - x - x^2\right) dx$ where $a \le b$. the boundary of the domain of F is the $a = b$ in the ab-plane, and $F(a,a) = 0$, so F is identically 0 on the boundary of its domain. For interior critical points we have:

$$\frac{\partial F}{\partial a} = -\left(6 - a - a^2\right) = 0 \Rightarrow -(3+a)(2-a) = 0 \Rightarrow a = -3, 2$$
$$\frac{\partial F}{\partial b} = -\left(6 - b - b^2\right) = 0 \Rightarrow -(3+b)(2-b) = 0 \Rightarrow b = -3, 2$$

Hence the candidadates for critical points are $(-3,-3)$, $(-3,2)$, $(2,-3)$, and $(2,2)$. Since $a \le b$, only one of these four points is a critical point, namely, it is $(-3,2)$. Next

$$F(-3,2) = \int_{-3}^{2} \left(6 - x - x^2\right) dx = \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2} = 12 - 2 - \frac{8}{3} - \left(-18 - \frac{9}{2} + 9\right) = 75/3$$

is the maximum value of F and gives the area under the parabola $y = 6 - x - x^2$ that is above the x-axis. Therefore $a = -3$ and $b = 2$.

p.695, pr.37

(b) 15 Points Find the absolute maximum and minimum values of $f(x,y) = x^2 + xy + y^2 - 3x + 3y$ on the given triangular region R.

Solution:
- *On AB*, we have $f(x,y) = f(x, 4-x) = x^2 - 10x + 28$ for $0 \le x \le 4$. So $f'(x) = 2x - 10 \Rightarrow x = 5 \notin [0,4] \Rightarrow$ no critical points in the interior of AB.

Endpoints of AB: $f(4,0) = 4$ and $f(0,4) = 28$.
- *On OB*, we have $f(x,y) = f(x,0) = x^2 - 3x$ for $0 \le x \le 4$. So $f'(x,0) = 2x - 3 = 0 \Rightarrow x = 3/2$ and $y = 0 \Rightarrow (3/2, 0)$ is an interior critical point of OB with $f(3/2, 0) = -9/4$.

Endpoints of BC: $f(0,0) = 0$ and $f(4,0) = 4$.
- *On OA*, we have $f(x,y) = f(0,y) = y^2 + 3y$ for $0 \le y \le 4$. So $f'(0,y) = 2y + 3 = 0 \Rightarrow y = -3/2$ and $x = 0 \Rightarrow (0, -3/2)$ is not on the segment OA. So no interior point occurs as C.P.

Endpoints of OA: $f(0,0) = 0$ and $f(0,4) = 28$.
- *Interior Points of this triangular region R*: $f_x(x,y) = 2x + y - 3 = 0$ and $f_y(x,y) = x + 2y + 3 = 0 \Rightarrow x = 3, y = -3 \Rightarrow (3,-3)$. But $(3,-3)$ is not an interior of R. So f has no interior critical point.

Therefore *abs. max* is 28 at $(0,4)$ and the *abs. min.* is $-9/4$ at $(3/2, 0)$.

p.695, pr.37