Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (a) 15 Points $\displaystyle\int \frac{\sqrt{9-w^2}}{w^2}\,dw = ?$

Solution:

$$\begin{aligned}
w &= 3\sin\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dw = 3\cos\theta\,d\theta,\ \sqrt{9-w^2} = 3\cos\theta;\\
\int \frac{\sqrt{9-w^2}}{w^2}\,dw &= \int \frac{3\cos\theta\,3\cos\theta\,d\theta}{9\sin^2\theta} = \int \cot^2\theta\,d\theta\\
&= \int\left(\frac{1-\sin^2\theta}{\sin^2\theta}\right)d\theta = \int(\csc^2\theta - 1)\,d\theta\\
&= -\cot\theta - \theta + C\\
&= -\frac{\sqrt{9-w^2}}{w} - \sin^{-1}\left(\frac{w}{3}\right) + C
\end{aligned}$$

p.452, pr.20

(b) 10 Points $\displaystyle\int \cos^3(2x)\sin^5(2x)\,dx = ?$

Solution:

$$\begin{aligned}
\int \cos^3(2x)\sin^5(2x)\,dx &= \frac{1}{2}\int \cos^3(2x)\sin^5(2x)\,2\,dx = \frac{1}{2}\int \cos(2x)\cos^2(2x)\sin^5(2x)\,2\,dx\\
&= \frac{1}{2}\int (1-\sin^2(2x))\sin^5(2x)\cos(2x)2\ dx\\
&= \frac{1}{2}\int \sin^5(2x)\cos(2x)\ 2\ dx - \frac{1}{2}\int \sin^7(2x)\cos(2x)\ 2\ dx\\
&= \frac{1}{12}\sin^6(2x) - \frac{1}{16}\sin^8(2x) + C
\end{aligned}$$

p.448, pr.12

2. (a) 7 Points Does the series $\displaystyle\sum_{n=2}^{\infty}\frac{\ln n}{n}$ converge? Give reasons.

Solution: The series diverges by the integral Test:

$$\int_2^{\infty}\frac{\ln x}{x}\,dx = \lim_{b\to\infty}\int_2^b \frac{\ln x}{x}\,dx = \lim_{b\to\infty}\frac{1}{2}\left(\ln^2 b - \ln^2 2\right) = \infty \Rightarrow \sum_{n=2}^{\infty}\frac{\ln n}{n} \text{ diverges.}$$

Alternatively, since $\ln x > 1$ for each $x > e$, we have $\dfrac{\ln n}{n} > \dfrac{1}{n}$, for each $n > 3$ and so $\displaystyle\sum_{n=3}^{\infty}\frac{\ln n}{n}$ diverges by the Direct Comparison Test (as $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n}$ is a divergent harmonic series). Therefore $\displaystyle\sum_{n=2}^{\infty}\frac{\ln n}{n}$ diverges too. p.557, pr.19

(b) 8 Points Does the series $\displaystyle\sum_{n=1}^{\infty}\frac{4^n}{(3n)^n}$ converge? Give reasons.

Solution:

First note that $\dfrac{4^n}{(3n)^n} \geq 0$ for all $n \geq 1$. We use the Root Test.

Since $\displaystyle\lim_{n\to\infty}\sqrt[n]{\frac{4^n}{(3n)^n}} = \lim_{n\to\infty}\left(\frac{4}{3n}\right) = 0 < 1 \Rightarrow \sum_{n=1}^{\infty}\frac{4^n}{(3n)^n}$ converges by the Root Test.

Alternatively, one can use the Ratio Test. For

$$\rho = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{4^{n+1}}{(3(n+1))^{n+1}}\cdot\frac{(3n)^n}{4^n} = \lim_{n\to\infty}\frac{4}{3}\frac{n^n}{(n+1)^n}\cdot\frac{1}{n+1} = \frac{4}{3}\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n\cdot\frac{1}{n+1} = \frac{4}{3}\cdot\frac{1}{e}\cdot 0 = 0 < 1$$

p.567, pr.11

(c) 10 Points Find the Taylor series generated by $f(x)=\dfrac{1}{x^2}$ at $a=1$ Give the interval of convergence for the series.

Solution: $f(x)=x^{-2}\Rightarrow f'(x)=-2x^{-3}, f''(x)=3!x^{-4}, f'''(x)=-4!x^{-5}\Rightarrow f^n(x)=(-1)^n(n+1)!x^{-n-2}$;
$f(1)=1$, $f'(1)=-2$, $f''(1)=3!$, $f'''(1)=-4!$, $f^n(1)=(-1)^n(n+1)!$

$\dfrac{1}{x^2}=1-2(x-1)+3(x-1)^2+4(x-1)^3+\cdots=\sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n.$

To find the interval of convergence, we use the Ratio Test.

$\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\dfrac{(-1)^{n+1}(n+2)(x-1)^{n+1}}{(-1)^n(n+1)(x-1)^n}\right|=|x-1|\lim_{n\to\infty}\dfrac{n+2}{n+1}=|x-1|<1\Leftrightarrow -1<x-1<1\Leftrightarrow 0<x<2$

Alternatively, we let $y=x-1$. Then $x=y+1$ and

$$\begin{aligned}\frac{1}{x} &= \frac{1}{y+1}=\sum_{n=0}^{\infty}(-y)^n\\ &= \sum_{n=0}^{\infty}[-(x-1)]^n\\ &= 1-(x-1)+(x-1)^2-(x-1)^3+(x-1)^4-\cdots\end{aligned}$$

(for $0<x<2$). Now

$$\begin{aligned}\frac{1}{x^2} &= \frac{1}{(y+1)^2}=-\frac{d}{dy}\frac{1}{1+y}=-\frac{d}{dy}\sum_{n=0}^{\infty}(-y)^n\\ &= -\sum_{n=1}^{\infty}n(-1)^n y^{n-1}\\ &= \sum_{n=1}^{\infty}n(-1)^{n+1}(x-1)^{n-1}=\sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n\end{aligned}$$

Again by Ratio Test the series converges absolutely for $0<x<2$.

p.567, pr.11

3. (a) 10 Points Suppose $f(x,y)=\tan^{-1}\left(\dfrac{x}{y}\right)$. Find $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}$ in its simplest form.

Solution:

$\dfrac{\partial f}{\partial x}=\dfrac{1/y}{1+\left(\frac{x}{y}\right)^2}=\dfrac{y}{y^2+x^2}$, $\dfrac{\partial f}{\partial y}=\dfrac{-x/y^2}{1+\left(\frac{x}{y}\right)^2}=\dfrac{-x}{y^2+x^2}$, $\dfrac{\partial^2 f}{\partial x^2}=\dfrac{1/y}{1+\left(\frac{x}{y}\right)^2}=\dfrac{(y^2+x^2)\cdot 0-y\cdot 2x}{(y^2+x^2)^2}=\dfrac{-2xy}{(y^2+x^2)^2}$,

$\dfrac{\partial^2 f}{\partial y^2}=\dfrac{1/y}{1+\left(\frac{x}{y}\right)^2}=\dfrac{(y^2+x^2)\cdot 0-(-x)\cdot 2y}{(y^2+x^2)^2}=\dfrac{2xy}{(y^2+x^2)^2}$,

$\Rightarrow \dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=\dfrac{1/y}{1+\left(\frac{x}{y}\right)^2}=\dfrac{(y^2+x^2)\cdot 0-y\cdot 2x}{(y^2+x^2)^2}=\dfrac{-2xy}{(y^2+x^2)^2}+\dfrac{2xy}{(y^2+x^2)^2}=0.$

p.774, pr.78

(b) 15 Points Use *only* the method of Lagrange multipliers to find the maximum and minimum values of $f(x,y,z)=x+2y+3z$ subject to $x^2+y^2+z^2=25$.

Solution:

$\nabla f=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ and $\nabla g=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$ so that $\nabla f=\lambda\nabla g\Rightarrow \mathbf{i}+2\mathbf{j}+3\mathbf{k}\lambda(2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k})\Rightarrow 1=2x\lambda$,
$2=2y\lambda$ and $3=2z\lambda\Rightarrow x=\dfrac{1}{2\lambda}$, $y=\dfrac{1}{\lambda}=2x$, and $z=\dfrac{3}{2\lambda}=3x$

$\Rightarrow x^2+(2x)^2+(3x)^2=25\Rightarrow x=\pm\dfrac{5}{\sqrt{14}}$.

Thus, $x=\dfrac{5}{\sqrt{14}}$, $y=\dfrac{10}{\sqrt{14}}$, $z=\dfrac{15}{\sqrt{14}}$ or

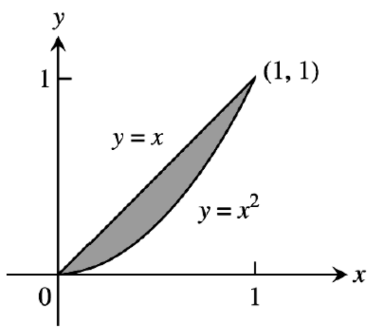
$x = -\dfrac{5}{\sqrt{14}}$ $y = -\dfrac{10}{\sqrt{14}}$, $z = -\dfrac{15}{\sqrt{14}}$. Therefore $f\left(\dfrac{5}{\sqrt{14}}, \dfrac{10}{\sqrt{14}}, \dfrac{15}{\sqrt{14}}\right) = 5\sqrt{14}$ is the maximum value and $f\left(-\dfrac{5}{\sqrt{14}}, -\dfrac{10}{\sqrt{14}}, -\dfrac{15}{\sqrt{14}}\right) = -5\sqrt{14}$ is the minimum value.

p.819, pr.24

4. (a) 15 Points Sketch the region and reverse the order of integration: $\int_0^1 \int_y^{\sqrt{y}} dx\,dy$. (DO NOT EVALUATE THE INTEGRAL)

Solution: The new double integral with the order reversed is

$$\int_0^1 \int_y^{\sqrt{y}} dx\,dy = \int_0^1 \int_{x^2}^{x} dy\,dx$$



p.848, pr.35

(b) 10 Points $\int_0^6 \int_{y^2/3}^{2y} dx\,dy = ?$

Solution:

$$\int_0^6 \int_{y^2/3}^{2y} dx\,dy = \int_0^6 \left(2y - \frac{y^2}{3}\right) dy = \left[y^2 - \frac{y^3}{9}\right]_0^6 = 36 - \frac{216}{9} = 12.$$

p.852, pr.13