Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 90 min.

Do not write in the table to the right.

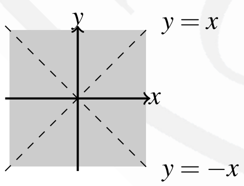
Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. Suppose $f(x,y) = \dfrac{xy+1}{x^2-y^2}$.

(a) $\boxed{\text{10 Points}}$ Sketch the domain of f.

Solution: For this rule f to make sense, we must exclude $\{(x,y)\in\mathbb{R}^2 \mid y^2 = x^2\}$. So the domain is

$$\boxed{D = \{(x,y)\in\mathbb{R}^2 \mid y \neq \pm x\}}$$



p.441, pr.8

(b) $\boxed{\text{10 Points}}$ Show that the limit $\lim\limits_{(x,y)\to(1,-1)} f(x,y)$ does not exist.

Solution: Along the path $y = -1$, we have

$$\begin{aligned}
\lim_{(x,y)\to(1,-1)} \frac{xy+1}{x^2-y^2} &= \lim_{(x,-1)\to(1,-1)} \frac{-x+1}{x^2-1} \\
&= \lim_{x\to 1} \frac{-1}{x+1} \\
&= \frac{-1}{2}
\end{aligned}$$

However, along the path $y = -x^2$, we have

$$\begin{aligned}
\lim_{(x,y)\to(1,-1)} \frac{xy+1}{x^2-y^2} &= \lim_{(x,-x^2)\to(1,-1)} \frac{-x^3+1}{x^2-x^4} \\
&= \lim_{x\to 1} \frac{\cancel{(1-x)}(1+x+x^2)}{\cancel{(1-x)}x^2(x+1)} \\
&= \lim_{x\to 1} \frac{1+x+x^2}{x^2(x+1)} \\
&= \frac{3}{2}
\end{aligned}$$

Since both limits do not agree, the original limit DOES NOT EXIST.

p.452, pr.24

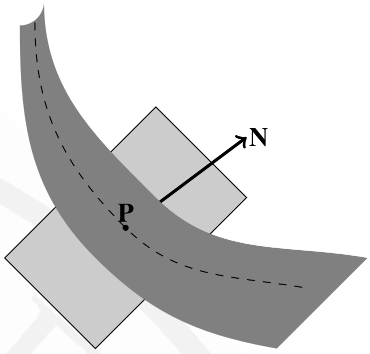
2. Given the surface $z = \dfrac{1}{x^2+3y^2}$ and the point $P_0(1,1,1/4)$ on it.

   (a) $\boxed{\text{7 Points}}$ Write the equation of the plane tangent to this surface at P_0.

Solution: Let $F(x,y,z) = \dfrac{1}{x^2+3y^2} - z$ so that $\nabla F(x,y,z) = \dfrac{-2x}{(x^2+3y^2)^2}\mathbf{i} + \dfrac{-6y}{(x^2+3y^2)^2}\mathbf{j} - \mathbf{k}$ and $\nabla F(1,1,1/4) = \dfrac{-2}{16}\mathbf{i} + \dfrac{-6}{16}\mathbf{j} - \mathbf{k}$. The tangent plane equation is then

$$-\frac{1}{8}(x-1) - \frac{3}{8}(y-1) - (z-\frac{1}{4}) = 0 \Rightarrow \boxed{x+3y+8z=6}$$

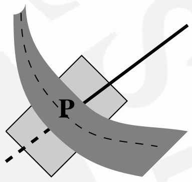
p.695, pr.23



   (b) $\boxed{\text{5 Points}}$ Write the parametric equations of the normal to this surface at P_0.

Solution: The normal has parametric equations $L: \begin{cases} x = 1 - \frac{1}{8}t, \\ y = 1 - \frac{3}{8}t, \\ z = \frac{1}{4} - t \end{cases}$

p.695, pr.37



   (c) $\boxed{\text{8 Points}}$ If $x = \cos(3t)$ and $y = \sin(3t)$, then $\left.\dfrac{dz}{dt}\right|_{t=0} = ?$

Solution:

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= \frac{-2x}{(x^2+3y^2)^2}\cdot(-3\sin(3t)) + \frac{-6y}{(x^2+3y^2)^2}\cdot(3\cos(3t)) \\
&= \frac{-2\cos(3t)}{(\cos^2(3t)+3\sin^2(3t))^2}\cdot(-3\sin(3t)) + \frac{-6\sin(3t)}{(\cos^2(3t)+3\sin^2(3t))^2}\cdot(3\cos(3t)) \\
\left.\frac{dz}{dt}\right|_{t=0} = &= \boxed{0}
\end{aligned}$$

p.695, pr.37

3. (a) $\boxed{\text{10 Points}}$ $\displaystyle\int \tan^{-1} x \, \mathrm{d}x = ?$

Solution: We integrate by parts. Let $u = \tan^{-1} x$ and so $dv = dx$. Then $du = \dfrac{1}{1+x^2}\,\mathrm{d}x$ and $v = x$. Therefore,

$$\begin{aligned}
\int \tan^{-1} x \, \mathrm{d}x &= \int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u \\
&= x\tan^{-1} x - \int \frac{x}{1+x^2}\,\mathrm{d}x \\
&= \boxed{x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C}
\end{aligned}$$

p.443, pr.68

   (b) $\boxed{\text{10 Points}}$ $\displaystyle\int_0^{\pi/4} \sin^2(2\theta)\cos^3(2\theta)\,\mathrm{d}\theta = ?$

Solution: Let $y = \sin(2\theta)$ and so $\mathrm{d}y = 2\cos(2\theta)\,\mathrm{d}\theta$. Now, when $\theta = 0$, we have $y = \sin 0 = 0$. When $x = \pi/4$, we have $y = \sin(\pi/2) = 1$. Then

$$\begin{aligned}
\int_0^{\pi/4} \sin^2(2\theta)\cos^3(2\theta)\,\mathrm{d}\theta &= \int_0^{\pi/4} \sin^2(2\theta)\cos^3(2\theta)\,\mathrm{d}\theta \\
&= \int_0^{\pi/4} \sin^2(2\theta)\cos^2(2\theta)\cos(2\theta)\,\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{\pi/4} \sin^2(2\theta)(1-\sin^2(2\theta))2\cos(2\theta)\,\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{1} y^2(1-y^2)\,\mathrm{d}y = \frac{1}{2}\int_0^{1}(y^2-y^4)\,\mathrm{d}y \\
&= \frac{1}{2}\left[\frac{y^3}{3}-\frac{y^5}{5}\right]_0^1 = \frac{1}{2}\left[\frac{1}{3}-\frac{1}{5}\right] = \boxed{\frac{1}{15}}
\end{aligned}$$

p.448, pr.22

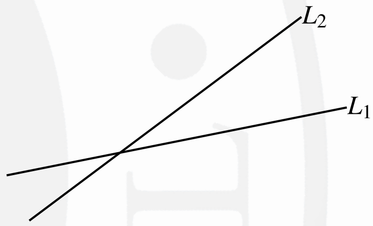
4. Two lines $L1 : \begin{cases} x = -1+t, \\ y = 2+t, \\ z = 1-t \end{cases}$ and $L2 : \begin{cases} x = 1-4s, \\ y = 1+2s, \\ z = 2-2s \end{cases}$ are given.

(a) $\boxed{\text{7 Points}}$ Find the point of intersection of $L1$ and $L2$.

Solution: To determine where $L1$ and $L2$ intersect, we solve simultaneously $-1+t = 1-4s$ and $2+t = 1+2s$ and we find the unique solution $t = 0$, $s = 1/2$. When we plug $t = 0$ in $z = 1-t$ we get $z = 1$. Similarly, if we plug $s = 1/2$ in $z = 2-2s$, we get the same $z = 1$ which means they both agree to imply that

$$\boxed{L1 \cap L2 = \{(1,2,1)\}}.$$

p.695, pr.29



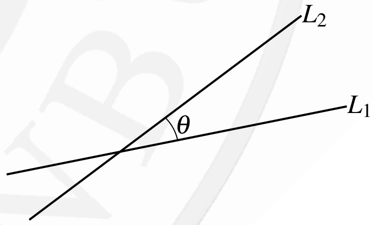
(b) $\boxed{\text{6 Points}}$ Find θ, if θ is the angle between $L1$ and $L2$.

Solution: First notice that the vector $\mathbf{v_1} = \mathbf{i}+\mathbf{j}-\mathbf{k}$ is parallel to $L1$ and $\mathbf{v_2} = -4\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ is parallel to $L2$. We know that θ is also the angle between $\mathbf{v_1}$ and $\mathbf{v_2}$. Hence

$$\begin{aligned}
\cos\theta = \frac{\mathbf{v_1}\cdot\mathbf{v_2}}{|\mathbf{v_1}||\mathbf{v_2}|} &= \frac{(1)(-4)+(1)(2)+(-1)(-2)}{\sqrt{(-4)^2+(2)^2+(-2)^2}\sqrt{(1)^2+(1)^2+(-1)^2}} \\
&= \frac{0}{\sqrt{24}\sqrt{3}} = 0
\end{aligned}$$

Therefore $\cos\theta = 0 \Rightarrow \theta = \pi/2$.

p.385, pr.88



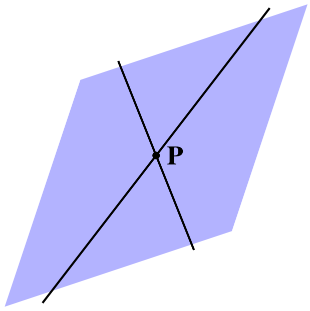
(c) $\boxed{\text{7 Points}}$ Write the equation of the plane determined by $L1$ and $L2$.

Solution: We shall write the equation of the plane through $(1,2,1)$ and admitting the vector $\mathbf{n} = \mathbf{v_1} \times \mathbf{v_2}$ as a normal vector. Now we have

$$\begin{aligned} \mathbf{n} = \mathbf{v_1} \times \mathbf{v_2} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ -4 & 2 & -2 \end{vmatrix} \\ &= 0\mathbf{i} + 6\mathbf{j} + 6\mathbf{k} \end{aligned}$$

This implies that $0(x-1)+6(y-2)+6(z-1)=0$ and so the equation of the desired plane is $\boxed{y+z=3}$.

p.385, pr.88



5. (a) $\boxed{\text{7 Points}}$ $\lim_{n\to\infty}\left(\frac{n-5}{n}\right)^n = ?$

Solution: This limit has the indeterminate 1^∞. Let $y := \left(\frac{x-5}{x}\right)^x = \left(1-\frac{5}{x}\right)^x$. Then

$\ln y = x\ln\left(1-\frac{5}{x}\right) = \frac{\ln(1-\frac{5}{x})}{1/x}$.

Now

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln(1-\frac{5}{x})\to\ln 1 = 0}{1/x\to 0}\to\frac{0}{0}.$$

By the L'Hôpital's Rule, we have

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\frac{1}{1-\frac{5}{x}}\left(\frac{5}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{-5}{1-\frac{5}{x}} = -5.$$

Hence $\ln y\to -5$ as $x\to\infty$. Therefore $y\to e^{-5}$ as $x\to\infty$. This shows that $\lim_{n\to\infty}\left(\frac{n-5}{n}\right)^n = \boxed{e^{-5}}$.

p.368, pr.42

(b) $\boxed{\text{5 Points}}$ Does the series $\sum_{n=1}^{\infty}\left(\frac{n-5}{n}\right)^n$ converge? Justify your answer.

Solution: Let $a_n = \left(\frac{n-5}{n}\right)^n$. By part (a), we have $\lim_{n\to\infty} a_n = e^{-5}\neq 0$. By the nth term test, the series *diverges*.

p.376, pr.30

(c) $\boxed{\text{8 Points}}$ Does the series $\sum_{n=2}^{\infty}\frac{1}{n\sqrt{n^2-1}}$ converge or diverge? Justify your answer.

Solution: We use the Limit Comparison Test. For each $n = 1,2,\cdots$, let $a_n = \frac{1}{n\sqrt{n^2-1}} > 0$ and $b_n = \frac{1}{n^2} > 0$. Then

$$c = \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{n\sqrt{n^2-1}}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^2}{\sqrt{n^2-1}} = \lim_{n\to\infty}\frac{\cancel{n^2}}{\cancel{n^2}\sqrt{1-\frac{1}{n^2}}} = \lim_{n\to\infty}\frac{1}{\sqrt{1-\frac{1}{n^2}}} = \boxed{1}$$

So $0 < c = 1 < \infty$ and $\sum_{n=2}^{\infty}\frac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$), the Limit Comparison Test yields that the given series *converges*.

Alternatively,one can use the (Direct) Comparison Test. For this, notice that $n^2-1 > \frac{1}{4}n^2$ for each $n = 2,3,\cdots$. Hence $\sqrt{n^2-1} > \frac{1}{2}n$ for $n = 2,3,\cdots$.

$$0 < a_n = \frac{1}{n\sqrt{n^2-1}} < \frac{2}{n^2}, \qquad n = 2,3,\cdots$$

Now $\sum_{n=2}^{\infty}\frac{2}{n^2}$ converges as it is a constant multiple of the convergent p-series $\sum_{n=2}^{\infty}\frac{1}{n^2}$ with ($p = 2 > 1$). Therefore, the given series does converge by the Comparison Test.

p.606, pr.40