Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place [a box around your answer] to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 75 min.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (a) [10 Points] Determine if the series $\sum_{n=2}^{\infty} \frac{n^2}{e^n}$ converges or diverges. Explain your answer.

Solution: Here one can use different tests.

(1) Ratio Test: Let $a_n = \frac{n^2}{e^n} > 0$ and so, $a_{n+1} = \frac{(n+1)^2}{e^{n+1}}$. Then we have

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^2}{e^{n+1}}}{\frac{n^2}{e^n}} = \lim_{n\to\infty} \frac{(n+1)^2}{\cancel{e^n} e} \frac{\cancel{e^n}}{n^2} = \frac{1}{e} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 = \frac{1}{e}.$$

Since $\rho < 1$, series converges by Ratio test.

(2) Direct Comparison Test: Notice for $n \geq 5$, we have $n^2 \leq 2^n$. Hence

$$0 < \frac{n^2}{e^n} \leq \frac{2^n}{e^n} = \left(\frac{2}{e}\right)^n.$$

The latter series $\sum_{n=2}^{\infty} (2/e)^n$ is a convergent geometric series (as $|r| = 2/e < 1$). Therefore the original series *converges*.

(3) Root Test: Here $a_n = \frac{n^2}{e^n} > 0$. Then $\sqrt[n]{a_n} = \sqrt[n]{\frac{n^2}{e^n}} = \frac{n^{2/n}}{e}$. Therefore

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{n^{2/n}}{e} = \frac{1}{e}\left(\lim_{n\to\infty} n^{1/n}\right)^2 = \frac{1}{e}(1)^2 = \frac{1}{e}.$$

Since $\rho = \frac{1}{e} < 1$, series converges by Root test.

(4) Limit Comparison Test: Let $a_n = \dfrac{n^2}{e^n} > 0$ and $b_n = \dfrac{1}{n^2} > 0$. Then four applications of L'Hôpital's Rule yields

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2}{e^n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4}{e^n} = \lim_{n\to\infty} \frac{4n^3}{e^n} = \lim_{n\to\infty} \frac{12n^2}{e^n} \lim_{n\to\infty} \frac{24n}{e^n} \lim_{n\to\infty} \frac{24}{e^n} = 0$$

Since this limit equals zero and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series (with $p = 2 > 1$), we see that $\sum a_n$ converges too.

(5) Integral Test: Let $f(x) = x^2 e^{-x}$ for $x \geq 2$. Then $f(n) = a_n$ for all $n \geq 2$. Notice that on $[2, \infty)$, $f(x)$ is

- continuous
- positive and
- decreasing as $f'(x) = -x^2e^{-x} + 2xe^{-x} = xe^{-x}(2-x) < 0$ for > 2.

Thus, the Integral Iest applies. Integrating twice by parts, we have

$$\int_2^{\infty} x^2 e^{-x}\,\mathrm{d}x = \lim_{b\to\infty} \int_2^b x^2 e^{-x}\,\mathrm{d}x = \lim_{b\to\infty} \Big[-x^2e^{-x} - 2xe^{-x} + 2e^{-x}\Big]_2^b$$
$$= \lim_{b\to\infty} \left[-\frac{b^2}{e^b} - \frac{2b}{e^b} + \frac{2}{e^b} - 6e^{-2}\right] = -6e^{-2}$$

Therefore this improper integral converges. Hence by the Integral Test, series must converge.

p.567, pr.18

(b) 15 Points Evaluate the integral $\displaystyle\int \frac{1}{(x^{1/3}-1)\sqrt{x}}\,\mathrm{d}x$. (*Hint*: Let $x = u^6$.)

Solution: Since $x = u^6$, we have $\mathrm{d}x = 6u^5\,\mathrm{d}u$. Then $x^{1/3} = (u^6)^{1/3} = u^2$ and $\sqrt{x} = \sqrt{u^6} = u^3$. Therefore

$$\int \frac{1}{(x^{1/3}-1)\sqrt{x}}\,\mathrm{d}x = \int \frac{6}{(u^2-1)\cancel{u^3}}\cancel{u^3}u^2\,\mathrm{d}u = 6\int \frac{u^2}{u^2-1}\,\mathrm{d}u = \int \left(6 + \frac{6}{u^2-1}\right)\mathrm{d}u$$

For the latter summand in the integrand, we use partial fraction decomposition.

$$\frac{6}{u^2-1} = \frac{6}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}$$
$$\Rightarrow A(u+1) + B(u-1) = 6$$
$$\Rightarrow \begin{cases} A + B = 0 \\ A - B = 6 \end{cases} \Rightarrow \boxed{A = 3}, \boxed{B = -3}$$
$$\Rightarrow \frac{6}{(u-1)(u+1)} = \frac{3}{u-1} - \frac{3}{u+1}$$
$$\Rightarrow \int \left(6 + \frac{6}{u^2-1}\right)\mathrm{d}u = \int \left(6 + \frac{3}{u-1} - \frac{3}{u+1}\right)\mathrm{d}u = 6u + 3\ln|u-1| - 3\ln|u+1| + C$$
$$= \boxed{6x^{1/6} + 3\ln|x^{1/6}-1| - 3\ln|x^{1/6}+1| + C}$$

p.462, pr.46

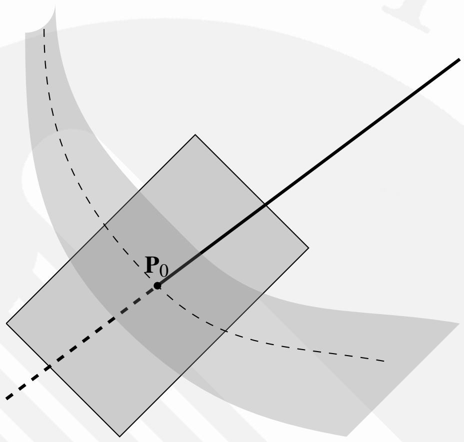
2. (a) [11 Points] Find parametric equations for the line normal to the surface $x^2z - y^2x + 3y - z = -4$ at $P_0(1,-1,2)$.

Solution: Let $F(x,y,z) = x^2z - y^2x + 3y - z + 4$. Then

$$F_x = 2xz - y^2 \Rightarrow F_x(1,-1,2) = 2(1)(2) - (-1)^2 = 3$$
$$F_y = -2yx + 3 \Rightarrow F_y(1,-1,2) = -2(-1)(1) + 3 = 5$$
$$F_z = x^2 - 1 \Rightarrow F_z(1,-1,2) = (1)^2 - 1 = 0$$

Thus the normal line will be parallel to the vector $\nabla F(1,-1,2) = 3\mathbf{i} + 5\mathbf{j} + 0\mathbf{k} = \boxed{3\mathbf{i} + 5\mathbf{j}}$ Therefore the parametric equations are

$$\mathscr{L} : \begin{cases} x = 1 + 3t \\ y = -1 + 5t \\ z = 2 \end{cases}$$



p.583, pr.32

(b) [14 Points] Find the local maxima and minima and saddle points for $f(x,y) = x^3 + y^3 - 3xy + 15$. Find function's value at these points.

Solution:

$$f_x = 3x^2 - 3y = 0 \qquad f_y = 3y^2 - 3x = 0$$
$$3(y^2)^2 - 3y = 0 \qquad 3y^2 = 3x$$
$$3y^4 - 3y = 0 \qquad x = y^2$$
$$3y(y^3 - 1) = 0$$
$$y = 0 \quad y = 1$$
$$x = 0 \quad x = 1$$

The critical points for this function are $(0,0)$ and $(1,1)$. Now we have

$$f_{xx} = 6x, \qquad f_{yy} = 6y, \qquad f_{xy} = -3, \qquad f_{xx}f_{yy} - (f_{xy})^2 = (6x)(6y) - (-3y)^2 = 36xy - 9y^2.$$

At $(0,0)$, we have

$$f_{xx}(0,0)f_{yy}(0,0) - (f_{xy}(0,0))^2 = (6(0))(6(0)) - (-3)^2 = 36(0)(0) - 9 = -9 < 0.$$

So f has a saddle point at $(0,0)$ and $f(0,0) = 15$.
At $(1,1)$, we have

$$f_{xx}(1,1)f_{yy}(1,1) - (f_{xy}(1,1))^2 = (6(1))(6(1)) - (-3)^2 = 36(1)(1) - 9 = 27 > 0 \quad \text{and} \quad f_{xx}(1,1) = 6 > 0$$

So f has a local minimum at $(1,1)$ and $f(1,1) = 14$.

p.588, pr.4

3. (a) 12 Points Find $\left.\frac{dw}{dt}\right|_{t=1}$ if $w = xe^y + y\sin z - \cos z$, $x = 2\sqrt{t}$, $y = t - 1 + \ln t$, and $z = \pi t$.

Solution: We employ the chain rule formula

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

Now the derivatives we require are

$$\frac{\partial w}{\partial x} = e^y = e^{t-1+\ln t} = te^{t-1}$$
$$\frac{\partial w}{\partial y} = xe^y + \sin z = (2\sqrt{t})te^{t-1} + \sin(\pi t) = 2t^{3/2}e^{t-1} + \sin(\pi t)$$
$$\frac{\partial w}{\partial z} = y\cos z + \sin z = (t-1+\ln t)\cos(\pi t) + \sin(\pi t)$$

and are

$$\frac{dx}{dt} = \frac{d}{dt}(2\sqrt{t}) = \frac{1}{\sqrt{t}}$$
$$\frac{dy}{dt} = \frac{d}{dt}(t-1+\ln t) = 1 + \frac{1}{t}$$
$$\frac{dy}{dt} = \frac{d}{dt}(\pi t) = \pi.$$

Hence

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = (e^y)\left(\frac{1}{\sqrt{t}}\right) + (xe^y + \sin z)\left(1 + \frac{1}{t}\right) + (y\cos z + \sin z)(\pi)$$

When $t = 1$, we have $x = 2$, $y = 0$, and $z = \pi$

$$\left.\frac{dw}{dt}\right|_{t=1} = \left(e^0\right)\left(\frac{1}{\sqrt{1}}\right) + \left(2e^0 + \sin\pi\right)\left(1 + \frac{1}{1}\right) + (0\cos\pi + \sin\pi)(\pi) = 1 + 2\cdot 2 = \boxed{5}$$

p.830, pr.30

(b) 13 Points Find an equation for the plane through $A(1,1,-1)$, $B(2,0,2)$, and $C(0,-2,1)$.

Solution: First we find a normal vector to the plane:

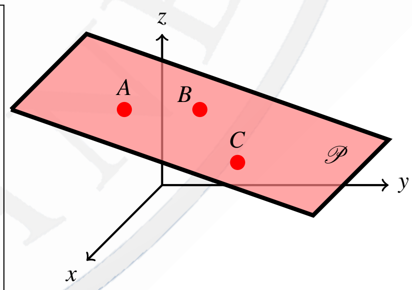
$$\begin{aligned}
\vec{AB} &= (2-1)\mathbf{i} + (0-1)\mathbf{j} + (2-(-1))\mathbf{k} \\
&= \mathbf{i} - \mathbf{j} + 3\mathbf{k} \\
\vec{AC} &= (0-1)\mathbf{i} + (-2-1)\mathbf{j} + (1-(-1))\mathbf{k} \\
&= -\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} \\
\Rightarrow \quad \vec{AB}\times\vec{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix} \\
&= 7\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}
\end{aligned}$$

is normal to the plane

$$\Rightarrow 7(x-2) - 5(y-0) - 4(z-2) = 0$$

hence $\boxed{7x - 5y - 4z = 6}$ is the equation of the plane.

p.695, pr.23



4. (a) 11 Points $\displaystyle\int_1^2 \frac{8\,\mathrm{d}x}{x^2-2x+2} = ?$

Solution: First notice that $x^2-2x+2=(x-1)^2+1$. Let $u=x-1$ and so $du=dx$. When $x=1$, we have $u=0$ and when $x=2$, we have $u=1$. Hence we have

$$\int_1^2 \frac{8\,\mathrm{d}x}{x^2-2x+2} = \int_1^2 \frac{8\,\mathrm{d}x}{(x-1)^2+1} = \int_0^1 \frac{8\,\mathrm{d}u}{u^2+1} = 8\left[\tan^{-1}u\right]_0^1 = 8\left(\tan^{-1}(1)-\tan^{-1}(0)\right)$$
$$= 8\left(\frac{\pi}{4}\right) = \boxed{2\pi}$$

p.487, pr.6

(b) 14 Points Does the limit $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ xy\neq 0}} \frac{x^2+y^2}{xy}$ exist? Why? Explain your answer.

Solution: The substitution $y=mx$ yields

$$\lim_{\substack{(x,y)\to(0,0)\\ xy\neq 0}} \frac{x^2+y^2}{xy} = \lim_{\substack{(x,mx)\to(0,0)\\ m\neq 0}} \frac{x^2+(mx)^2}{x(mx)} = \lim_{x\to 0} \frac{\cancel{x^2}(1+m^2)}{\cancel{x^2}(m)} = \lim_{x\to 0} \frac{1+m^2}{m}$$
$$= \frac{1+m^2}{m}$$

This is the limit as $(x,y)\to(0,0)$ along the straight line of slope $m\neq 0$. Different values of $m\neq 0$ (such as 1 and -1) give different values for the limit. Hence if $(x,y)\to(0,0)$, along the line $y=x$ (where $m=1$) the limit is $\frac{1+1^2}{1}=2$, whereas if $(x,y)\to(0,0)$ along the line $y=-x$ (where $m=-1$) the limit is $\frac{1+(-1)^2}{(-1)}=-2$. Therefore *the given limit does not exist*.

p.830, pr.16