Math 114 Summer 2017 **Final Exam** August 4, 2017

Your Name / Ad - Soyad ______ (**90 min.**) Signature / İmza ______

Student ID # / Öğrenci No ______ (**mavi tükenmez!**)

Problem	1	2	3	4	Total
Points:	27	20	28	25	100
Score:					

You have **90 minutes**. **(Cell phones off and away!)**. No books, notes or calculators are permitted. **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

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1. (a) (9 Points) Does the series $\sum_{n=1}^{\infty}\left(\frac{3n+1}{2n+1}\right)^n$ converge? Give reasons for your answer.

○ Converges. ○ Diverges. Test Used: ______

Solution: Here $a_n = \left(\frac{3n+1}{2n+1}\right)^n > 0$. Use the Root Test.

$$\rho = \lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{3n+1}{2n+1}\right)^n}$$
$$= \lim_{n\to\infty}\frac{3n+1}{2n+1} = \lim_{n\to\infty}\frac{3+1/n}{2+1/n} = \frac{3+0}{2+0} = \frac{3}{2} > 1$$

Hence by the Ratio Test, the series *diverges*.

p.491, pr.65

(b) (10 Points) Evaluate the integral $\int_0^{\infty}\frac{16\tan^{-1}x}{1+x^2}\,\mathrm{d}x$.

○ Converges. ○ Diverges. Integral's value = ______

Solution: If we let $u = \tan^{-1}x$, then we have $du = \frac{1}{1+x^2}\,\mathrm{d}x$. Therefore

$$\int\frac{16\tan^{-1}x}{1+x^2}\,\mathrm{d}x = \int 16u\,\mathrm{d}u = 8u^2 + C.$$
$$= 8\left(\tan^{-1}x\right)^2 + C$$

Then we have

$$\int_0^{\infty}\frac{16\tan^{-1}x}{1+x^2}\,\mathrm{d}x = \lim_{b\to\infty}\int_0^{b}\frac{16\tan^{-1}x}{1+x^2}\,\mathrm{d}x$$
$$= \lim_{b\to\infty}\left[8\left(\tan^{-1}x\right)^2\right]_0^b$$
$$= \lim_{b\to\infty}\left[8\left(\tan^{-1}b\right)^2 - 8\left(\tan^{-1}0\right)^2\right]$$
$$= 8\frac{\pi^2}{16} = \boxed{\frac{\pi^2}{2}}$$

Therefore the improper integral *converges* and has value $\frac{\pi^2}{2}$.

p.491, pr.86

(c) (8 Points) Does the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\tan^{-1} n}{n^2+1}$ converge absolutely? Give reason for your answer. (*Hint*: Use part (b).)

◯ Converges absolutely. ◯ Converges conditionally. ◯ Diverges. Test Used: ____________

Solution: The series *converges absolutely*, since the corresponding improper integral

$$\begin{aligned}\int_1^{\infty}\frac{\tan^{-1}x}{1+x^2}\,\mathrm{d}x &= \lim_{b\to\infty}\int_1^{b}\frac{\tan^{-1}x}{1+x^2}\,\mathrm{d}x\\ &= \lim_{b\to\infty}\left[\frac{\left(\tan^{-1}x\right)^2}{2}\right]_1^b\\ &= \frac{1}{2}\lim_{b\to\infty}\left[\left(\tan^{-1}b\right)^2-\left(\tan^{-1}1\right)^2\right]\\ &= \boxed{3\frac{\pi^2}{32}}\end{aligned}$$

converges.

p.491, pr.107

2. Given the point $Q(1,-1,5)$ and the line $\mathscr{L}:\begin{cases}x=1+2t,\\ y=-1+3t,\\ z=4+t\end{cases}$.

(a) (10 Points) Find the distance from the point Q to the line $\mathscr{L}$.

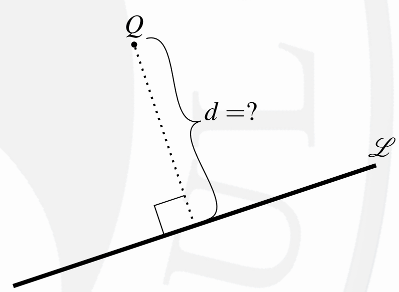
Solution: We shall use the distance formula $d=\dfrac{|\vec{PQ}\times\mathbf{v}|}{|\mathbf{v}|}$. Here (by letting $t=0$) $P(1,-1,4)$ is a point on $\mathscr{L}$ and $\mathbf{v}=2\mathbf{i}+3\mathbf{j}+\mathbf{k}$ is a vector that is parallel to $\mathscr{L}$. Now we have $\vec{PQ}=0\mathbf{i}+0\mathbf{j}+\mathbf{k}$ and so

$$\begin{aligned}\vec{PQ}\times\mathbf{v} &= \begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&1\\2&3&1\end{vmatrix} = \mathbf{i}\begin{vmatrix}0&1\\3&1\end{vmatrix}-\mathbf{j}\begin{vmatrix}0&1\\2&1\end{vmatrix}+\mathbf{k}\begin{vmatrix}0&0\\2&3\end{vmatrix}\\ &= -3\mathbf{i}+2\mathbf{j}+0\mathbf{k}\end{aligned}$$

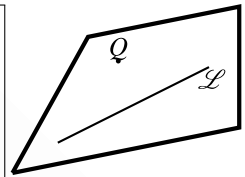
Therefore, we have

$$d=\frac{|\vec{PQ}\times\mathbf{v}|}{|\mathbf{v}|}=\frac{\sqrt{(-3)^2+(2)^2+(0)^2}}{\sqrt{(2)^2+(3)^2+(1)^2}}=\boxed{\frac{\sqrt{13}}{\sqrt{14}}}$$

p.695, pr.37



(b) (10 Points) Find the equation of the plane which contains both the point Q and the line $\mathscr{L}$.



Solution: We know from part (a) that the points $P(1,-1,4)$ and $Q(1,-1,5)$ are on the plane. Setting $t=1$, we get another point $R(3,2,5)$ which must also lie on the plane. Let $\mathbf{a}$ be the vector from $R(3,2,5)$ to $P(1,-1,4)$;

$$\mathbf{a} = (1-3)\mathbf{i} + (-1-2)\mathbf{j} + (4-5)\mathbf{k} = -2\mathbf{i} - 3\mathbf{j} - \mathbf{k}.$$

Let $\mathbf{b}$ be the vector from $R(3,2,5)$ to $Q(1,-1,5)$;

$$\mathbf{b} = (1-3)\mathbf{i} + (-1-2)\mathbf{j} + (5-5)\mathbf{k} = -2\mathbf{i} - 3\mathbf{j} + 0\mathbf{k}.$$

A normal vector $\mathbf{n}$ for the plane may be found by means of cross products.

$$\mathbf{n} = \mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -3 & -1 \\ -2 & -3 & 0 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -3 & -1 \\ -3 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -2 & -1 \\ -2 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -2 & -3 \\ -2 & -3 \end{vmatrix}$$
$$= -3\mathbf{i} + 2\mathbf{j} - 0\mathbf{k}$$

The general equation of a plane, as we know, is:

$$\underbrace{(-3\mathbf{i}+2\mathbf{j}-0\mathbf{k})}_{\mathbf{n}} \cdot \underbrace{((x-1)\mathbf{i}+(y+1)\mathbf{j}+(z-5)\mathbf{k})} = 0$$
$$\Rightarrow 3(x-1) - 2(y+1) - 0(z-5) = 0$$
$$\Rightarrow \boxed{3x-2y=5}$$

p.695, pr.37

3. (a) (10 Points) Write parametric equations for the line normal to the surface $\cos(\pi x) - x^2y + e^{xz} + yz = 4$ at the point $P_0(0,1,2)$.

Solution: We need a vector that is parallel to the normal line and it is

$$\begin{aligned} \nabla f &= f_x\,\mathbf{i} + f_y\,\mathbf{j} + f_z\,\mathbf{k} \\ &= (-\pi\sin(\pi x) - 2xy + ze^{xz})\,\mathbf{i} + \left(-x^2+z\right)\mathbf{j} + (xe^{xz}+y)\,\mathbf{k} \\ \Rightarrow &\nabla f(0,1,2) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \end{aligned}$$

Then the parametric equations for the normal line are $\mathscr{L}: \begin{cases} x = 2t, \\ y = 1+2t, \text{ for } -\infty < t < \infty \\ z = 2+t \end{cases}$.

p.72, pr.8

(b) (9 Points) Suppose $z = f(u,v)$, $u = x^2 - y^2$, $v = 2xy$, where f is a differentiable function. Show that

$$x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} = 2(x^2+y^2)\frac{\partial f}{\partial u}.$$

Solution: By using the Chain Rule, we have

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \\ &= \left(\frac{\partial f}{\partial u}\right)(2x) + \left(\frac{\partial f}{\partial v}\right)(2y) = 2x\frac{\partial f}{\partial u} + 2y\frac{\partial f}{\partial v} \\ \Rightarrow x\frac{\partial f}{\partial x} &= x\left(2x\frac{\partial f}{\partial u} + 2y\frac{\partial f}{\partial v}\right) = 2x^2\frac{\partial f}{\partial u} + 2xy\frac{\partial f}{\partial v} \end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} \\ &= \left(\frac{\partial f}{\partial u}\right)(-2y) + \left(\frac{\partial f}{\partial v}\right)(2x) = -2y\frac{\partial f}{\partial u} + 2x\frac{\partial f}{\partial v} \\ &\Rightarrow -y\frac{\partial f}{\partial y} = -y\left(-2y\frac{\partial f}{\partial u} + 2x\frac{\partial f}{\partial v}\right) = 2y^2\frac{\partial f}{\partial u} - 2xy\frac{\partial f}{\partial v}\end{aligned}$$

Hence we have

$$x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} = 2x^2\frac{\partial f}{\partial u} + \cancel{2xy\frac{\partial f}{\partial v}} + 2y^2\frac{\partial f}{\partial u} - \cancel{2xy\frac{\partial f}{\partial v}} = 2x^2\frac{\partial f}{\partial u} + 2y^2\frac{\partial f}{\partial u} = \left(2x^2 + 2y^2\right)\frac{\partial f}{\partial u}$$

p.317, pr.33

(c) (9 Points) Find the smallest directional derivative of

$$h(x,y) = x^3 + x^2y + y^2$$

at $(x,y) = (1,2)$. In which direction does this derivative occur?

Solution:

$$\begin{aligned}\nabla h(x,y) &= h_x\,\mathbf{i} + h_y\,\mathbf{j} \\ &= (3x^2 + 2xy)\,\mathbf{i} + (x^2 + 2y)\,\mathbf{j} \\ \nabla h(1,2) &= 7\,\mathbf{i} + 5\,\mathbf{j} \Rightarrow |\nabla h(1,2)| = \sqrt{7^2 + 5^2} = \sqrt{74}\end{aligned}$$

Smallest directional derivative has value $= \boxed{-\sqrt{74}}$ and it occurs in the direction $\boxed{-\frac{7}{\sqrt{74}}\,\mathbf{i} - \frac{5}{\sqrt{74}}\,\mathbf{j}}$

p.317, pr.33

4. (a) (11 Points) Find all the local maxima, local minima, and saddle points of the function $f(x,y) = x^2y - 2x^2 - 2y^2 + 4y$.

Solution:

$f_x = 2xy - 4x = 0,$

$(2x)(y-2) = 0$

$f_y = x^2 - 4y + 4 = 0$

$x = 0$ gives $-4y + 4 = 0$

$x = 2$ gives $x^2 - 4 = 0$ so $x = \pm 2$

$2x = (2x-2), 2y = (2y-4)$

Thus CP's are$(x,y) = (0,1), (2,2), (-2,2)$

$f_{xx} = 2y - 4$ $\quad f_{yy} = -4$ and $f_{xy} = f_{yx} = 2x$

Thus $D = (2y-4)(-4) - 4x^2$

$D(0,1) = (4)(2)$. Since also $f_{xx}(0,1) = -4 < 0$ Local max.

$D(2,2) = -16 < 0$. SADDLE POINT

$D(-2,2) = -16 < 0$. SADDLE POINT

p.95, pr.68

(b) (14 Points) Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x,y) = x^2 + y^2$ subject to the constraint $x^2 - 2x + y^2 - 4y = 0$.

Solution:

$f = x^2 + y^2,$ $g = x^2 - 2x + y^2 - 4y = 0$

$\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ $\nabla g = (2x-2)\mathbf{i} + (2y-4)\mathbf{j}$

$\nabla f = \lambda \nabla g \longrightarrow$ $2x = (2x-2)\lambda, 2y = (2y-4)\lambda$

$2x = 2x\lambda - 2\lambda$ $2y = 2y\lambda - 4\lambda$

$2x - 2x\lambda = -2\lambda$ $2y - 2y\lambda = -4\lambda$

$2x(1-\lambda) = -2\lambda$ $2y(1-\lambda) = -4\lambda$

$x = \dfrac{-\lambda}{1-\lambda}$ $y = \dfrac{-2\lambda}{1-\lambda}$

$$x^2 - 2x + y^2 - 4y = 0 \longrightarrow \frac{\lambda^2}{(1-\lambda)^2} + \frac{2\lambda}{1-\lambda} + \frac{4\lambda^2}{(1-\lambda)^2} + \frac{8\lambda}{1-\lambda} = 0$$

$$\frac{5\lambda^2}{(1-\lambda)^2} + \frac{10\lambda}{1-\lambda} = 0 \longrightarrow 5\lambda^2 + 10\lambda(1-\lambda) = 0$$

$5\lambda^2 - 10\lambda^2 + 10\lambda = 0 \longrightarrow -5\lambda^2 + 10\lambda = 0 \longrightarrow -5\lambda(\lambda - 2) = 0 \longrightarrow \lambda = 0$ or $\lambda = 2$

CASE I: $\lambda = 0 \longrightarrow x = 0$ and $y = 0$. $g(0,0) = 0$ is true. $f(0,0) = 0$ is a minimum.

CASE II: $\lambda = 2 \longrightarrow x = -2/(1-2) = 2$ and $y = -4/(1-2) = 4$. $g(2,4) = 4 - 4 + 16 - 16 = 0$ is true. $f(2,4) = 20$ is the maximum.

p.583, pr.39