Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- This exam has 3 pages plus this cover sheet and 4 problems. Please make sure that your exam is complete.
- Time limit is 90 min.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

Do not write in the table to the right.

1. (a) $\boxed{\text{9 points}}$ Evaluate the integral $\int x^2 \sin x\, dx$.

Solution: We choose $u = x^2$ and $dv = \sin x\, dx$. Then $du = 2x\, dx$ and $v = -\cos x$. With this choice, integration by parts yields

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2\int x \cos x\, dx.$$

For the last integral, we could do it using a further integration by parts. We now choose $u = x$ and so $dv = \cos x\, dx$. Then $du = dx$ and $v = \sin x$. Applying integration by parts to the last integral, we now have

$$\begin{aligned}\int x^2 \sin x\, dx &= -x^2 \cos x + 2\int x \cos x\, dx \\ &= -x^2 \cos x + 2\left(x \sin x - \int \sin x\, dx\right) \\ &= \boxed{-x^2 \cos x + 2x \sin x + 2\cos x + C.}\end{aligned}$$

p.441, pr.4

(b) $\boxed{\text{9 points}}$ Evaluate the integral $\int \frac{t^2+8}{t^2-5t+6}\, dt$.

Solution: $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{5t+2}{t^2-5t+6}$ (after long division); $\frac{5t+2}{t^2-5t+6} = \frac{5t+2}{(t-3)(t-2)} = \frac{A}{t-3} + \frac{B}{t-2} \Rightarrow$
$5t+2 = A(t-2) + B(t-3) = (A+B)t + (-2A-3B) \Rightarrow$

$$\left|\begin{array}{l} A+B=5 \\ -2A-3B=2 \end{array}\right\}$$

So $-B = (10+2) = 12 \Rightarrow B = -12 \Rightarrow A = 17$; thus, $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{17}{t-3} + \frac{-12}{t-2}$. Therefore $\int \frac{t^2+8}{t^2-5t+6} =$
$\int \left(1 + \frac{17}{t-3} + \frac{-12}{t-2}\right) dt = t + 17\ln|t-3| - 12\ln|t-2| + C.$ p.461, pr.7

(c) $\boxed{\text{7 points}}$ Determine if the integral $\int_0^\infty \frac{dx}{\sqrt{x^6+1}}$ converges or diverges. Explain your answer.

Solution:
$\int_0^\infty \frac{dx}{\sqrt{x^6+1}} = \int_0^1 \frac{dx}{\sqrt{x^6+1}} + \int_1^\infty \frac{dx}{\sqrt{x^6+1}} < \int_0^1 \frac{dx}{\sqrt{x^6+1}} + \int_1^\infty \frac{dx}{\sqrt{x^6}}$ and $\int_1^\infty \frac{dx}{x^3} = \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{2b^2} + \frac{1}{2}\right) =$
$\frac{1}{2} \Rightarrow \int_0^\infty \frac{dx}{\sqrt{x^6+1}}$ converges by the Direct Comparison Test. p.487, pr.51

2. (a) 8 points Does the series $\sum_{n=0}^{\infty}(-1)^n\frac{5}{4^n}$ converge? If so, find its sum.

Solution: $\sum_{n=0}^{\infty}(-1)^n\frac{5}{4^n}=5\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n$. The latter is geometric series with the first term $a=5$ and ratio $r=-\frac{1}{4}$. Since $|r|=\frac{1}{4}$, it converges to $\frac{a}{1-r}=\frac{5}{1+1/4}=4$.

(b) 9 points Determine if the series $\sum_{n=1}^{\infty}\frac{(n+3)!}{3!n!3^n}$ converges or diverges. Give reasons.

Solution: Use the Ratio test.

$$\rho=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+4)!}{3!(n+1)!3^{n+1}}\frac{3!n!3^n}{(n+3)!}=\lim_{n\to\infty}\frac{n+4}{3(n+1)}=\frac{1}{3}<1,$$

so the given series converges.

(c) 8 points Does $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converge absolutely, which converge, and which diverge? Give reasons.

Solution: The series has the form $\sum_{n=1}^{\infty}(-1)^n u_n$ where $u_n=\frac{1}{\sqrt{n}}$ is positive, decreasing and converges to 0. Hence $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test, but its series of absolute values $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a divergent p-series with $p=\frac{1}{2}$, so the given series converges conditionally.

3. (a) 15 points Find the radius and interval of convergence for $\sum_{n=0}^{\infty}\frac{n(x+3)^n}{5^n}$

Solution: If $a_n:=\frac{n(x+3)^n}{5^n}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)(x+3)^{n+1}}{5^{n+1}}\frac{5^n}{n(x+3)^n}\right|=\lim_{n\to\infty}\frac{1}{5}\frac{n+1}{n}|x+3|=\frac{|x+3|}{5}<1\Leftrightarrow|x+3|<5$$

so $[R=5]\Leftrightarrow -5<x+3<5\Leftrightarrow -8<x<2$. When $x=-8$, the series becomes $\sum_{n=1}^{\infty}\frac{n(-5)^n}{5^n}=\sum_{n=1}^{\infty}(-1)^n n$ which diverges by the n-th term test. When $x=2$, the series becomes $\sum_{n=1}^{\infty}\frac{n(5)^n}{5^n}=\sum_{n=1}^{\infty}n$ which also diverges by the n-th term test. Thus, the interval of convergence is $I=(-8,2)$.

(b) $\boxed{\text{10 points}}$ Find the Taylor series at $x = 0$ for $f(x) = \dfrac{x^2}{1-2x}$. Give the radius of convergence.

Solution: In the geometric series formula $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$ for $-1 < x < 1$, we replace x with $2x$. Hence

$$f(x) = \frac{x^2}{1-2x} = x^2 \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^{n+2} = x^2 + 2x^3 + 2^2x^4 + \cdots.$$

The series converges only when $|2x| < 1$; that is, when $|x| < \dfrac{1}{2}$, so the radius of convergence is $\boxed{R = \dfrac{1}{2}}$ and $I = \left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$.

4. (a) $\boxed{\text{8 points}}$ Find parametrization for the line segment joining the points $(0,1,1)$ and $(0,-1,1)$.

Solution: First, a vector that is parallel to the line is

$$\vec{PQ} = (0-0)\mathbf{i} + (-1-1)\mathbf{j} + (1-1)\mathbf{k} = -2\mathbf{j}.$$

Picking either point will give us equations for the line. Here, we use P, so that the parametric equations for the line are

$$\boxed{x = 0, \quad y = 1 - 2t, \quad z = 1, \quad 0 \le t \le 1.}$$

p.694, pr.17

(b) $\boxed{\text{8 points}}$ Find equations for the plane through $P_0(2,4,5)$ perpendicular to the line

$$x = 5 + t, y = 1 + 3t, z = 4t$$

Solution: We only need a normal vector $\mathbf{n}$ for the plane we want. The given line is parallel to the vector $\mathbf{v} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$. Hence we can take $\mathbf{n} = \mathbf{v} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$. Therefore the equation for the plane is

$$\begin{aligned}
\mathbf{n} \cdot \vec{P_0P} &= 0 \Rightarrow (1)(x-2) + (3)(y-4) + (4)(z-5) = 0 \\
&\Rightarrow x + 3y + 4z = 2 + 12 + 20 = 34 \\
&\Rightarrow \boxed{x + 3y + 4z = 34}.
\end{aligned}$$

p.694, pr.25

(c) $\boxed{\text{9 points}}$ Find the area of the triangle whose vertices are $A(1,-1,1)$, $B(0,1,1)$, and $C(1,0,-1)$.

Solution: We use the formula $\dfrac{1}{2}(|\vec{AB} \times \vec{AC}|)$.

$$\begin{aligned}
\vec{AB} &= (0-1)\mathbf{i} + (1-(-1))\mathbf{j} + (1-1)\mathbf{k} \\
&= -\mathbf{i} + 2\mathbf{j} \\
\vec{AC} &= (1-1)\mathbf{i} + (0-(-1))\mathbf{j} + (-1-1)\mathbf{k} \\
&= \mathbf{j} - 2\mathbf{k} \\
\Rightarrow \vec{AB} \times \vec{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ 0 & 1 & -2 \end{vmatrix} \\
&= -4\mathbf{i} - 2\mathbf{j} - \mathbf{k}
\end{aligned}$$

Therefore, the required area is

$$\frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{1}{2}\sqrt{(-4)^2 + (-2)^2 + (-1)^2} = \frac{1}{2}\sqrt{21}$$

p.687, pr.47