Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- This exam has 3 pages plus this cover sheet and 4 problems. Please make sure that your exam is complete.
- # Time limit is 90 min.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (a) [9 points] Evaluate the integral $\int x^2 \sin x\,dx$.

Solution: We choose $u = x^2$ and $dv = \sin x\,dx$. Then $du = 2x\,dx$ and $v = -\cos x$. With this choice, integration by parts yields

$$\int x^2 \sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx.$$

For the last integral, we could do it using a further integration by parts. We now choose $u = x$ and so $dv = \cos x\,dx$. Then $du = dx$ and $v = \sin x$. Applying integration by parts to the last integral, we now have

$$\begin{aligned}\int x^2 \sin x\,dx &= -x^2\cos x + 2\int x\cos x\,dx \\ &= -x^2\cos x + 2\left(x\sin x - \int \sin x\,dx\right) \\ &= \boxed{-x^2\cos x + 2x\sin x + 2\cos x + C.}\end{aligned}$$

p.441, pr.4

(b) [9 points] Evaluate the integral $\int \frac{t^2+8}{t^2-5t+6}\,dt$.

Solution: $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{5t+2}{t^2-5t+6}$ (after long division); $\frac{5t+2}{t^2-5t+6} = \frac{5t+2}{(t-3)(t-2)} = \frac{A}{t-3} + \frac{B}{t-2} \Rightarrow$ $5t+2 = A(t-2) + B(t-3) = (A+B)t + (-2A-3B) \Rightarrow$

$$\left|\begin{array}{r} A+B=5 \\ -2A-3B=2 \end{array}\right\}$$

So $-B = (10+2) = 12 \Rightarrow B = -12 \Rightarrow A = 17$; thus, $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{17}{t-3} + \frac{-12}{t-2}$. Therefore $\int \frac{t^2+8}{t^2-5t+6} =$ $\int\left(1 + \frac{17}{t-3} + \frac{-12}{t-2}\right)dt = t + 17\ln|t-3| - 12\ln|t-2| + C.$ p.461, pr.7

(c) [7 points] Determine if the integral $\int_0^\infty \frac{dx}{\sqrt{x^6+1}}$ converges or diverges. Explain your answer.

Solution:
$\int_0^\infty \frac{dx}{\sqrt{x^6+1}} = \int_0^1 \frac{dx}{\sqrt{x^6+1}} + \int_1^\infty \frac{dx}{\sqrt{x^6+1}} < \int_0^1 \frac{dx}{\sqrt{x^6+1}} + \int_1^\infty \frac{dx}{\sqrt{x^6}}$ and $\int_1^\infty \frac{dx}{x^3} = \lim_{b\to\infty}\left[-\frac{1}{2x^2}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{2b^2} + \frac{1}{2}\right) =$ $\frac{1}{2} \Rightarrow \int_0^\infty \frac{dx}{\sqrt{x^6+1}}$ converges by the Direct Comparison Test. p.487, pr.51

2. (a) [8 points] Does the series $\sum_{n=0}^{\infty}(-1)^n\frac{5}{4^n}$ converge? If so, find its sum.

Solution: $\sum_{n=0}^{\infty}(-1)^n\frac{5}{4^n}=5\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n$. The latter is geometric series with the first term $a=5$ and ratio $r=-\frac{1}{4}$. Since $|r|=\frac{1}{4}$, it converges to $\frac{a}{1-r}=\frac{5}{1+1/4}=4$.

(b) [9 points] Determine if the series $\sum_{n=1}^{\infty}\frac{(n+3)!}{3!n!3^n}$ converges or diverges. Give reasons.

Solution: Use the Ratio test.

$$\rho=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+4)!}{3!(n+1)!3^{n+1}}\frac{3!n!3^n}{(n+3)!}=\lim_{n\to\infty}\frac{n+4}{3(n+1)}=\frac{1}{3}<1,$$

so the given series converges.

(c) [8 points] Does $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converge absolutely, which converge, and which diverge? Give reasons.

Solution: The series has the form $\sum_{n=1}^{\infty}(-1)^n u_n$ where $u_n=\frac{1}{\sqrt{n}}$ is positive, decreasing and converges to 0. Hence $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test, but its series of absolute values $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a divergent p-series with $p=\frac{1}{2}$, so the given series converges conditionally.

3. (a) [15 points] Find the radius and interval of convergence for $\sum_{n=0}^{\infty}\frac{n(x+3)^n}{5^n}$

Solution: If $a_n:=\frac{n(x+3)^n}{5^n}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)(x+3)^{n+1}}{5^{n+1}}\frac{5^n}{n(x+3)^n}\right|=\lim_{n\to\infty}\frac{1}{5}\frac{n+1}{n}|x+3|=\frac{|x+3|}{5}<1\Leftrightarrow|x+3|<5$$

so $[R=5]\Leftrightarrow -5<x+3<5\Leftrightarrow -8<x<2$. When $x=-8$, the series becomes $\sum_{n=1}^{\infty}\frac{n(-5)^n}{5^n}=\sum_{n=1}^{\infty}(-1)^n n$ which diverges by the n-th term test. When $x=2$, the series becomes $\sum_{n=1}^{\infty}\frac{n(5)^n}{5^n}=\sum_{n=1}^{\infty}n$ which also diverges by the n-th term test. Thus, the interval of convergence is $I=(-8,2)$.

(b) 10 points Find the Taylor series at $x=0$ for $f(x)=\dfrac{x^2}{1-2x}$. Give the radius of convergence.

Solution: In the geometric series formula $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$ for $-1<x<1$, we replace x with $2x$. Hence

$$f(x)=\frac{x^2}{1-2x}=x^2\sum_{n=0}^{\infty}(2x)^n=\sum_{n=0}^{\infty}2^n x^{n+2}=x^2+2x^3+2^2x^4+\cdots.$$

The series converges only when $|2x|<1$; that is, when $|x|<\dfrac{1}{2}$, so the radius of convergence is $\boxed{R=\dfrac{1}{2}}$ and $I=\left(-\dfrac{1}{2},\dfrac{1}{2}\right)$.

4. (a) 8 points Find parametrization for the line segment joining the points $(0,1,1)$ and $(0,-1,1)$.

Solution: First, a vector that is parallel to the line is

$$\vec{PQ}=(0-0)\mathbf{i}+(-1-1)\mathbf{j}+(1-1)\mathbf{k}=-2\mathbf{j}.$$

Picking either point will give us equations for the line. Here, we use P, so that the parametric equations for the line are

$$\boxed{x=0,\quad y=1-2t,\quad z=1,\quad 0\le t\le 1.}$$

p.694, pr.17

(b) 8 points Find equations for the plane through $P_0(2,4,5)$ perpendicular to the line

$$x=5+t, y=1+3t, z=4t$$

Solution: We only need a normal vector $\mathbf{n}$ for the plane we want. The given line is parallel to the vector $\mathbf{v}=\mathbf{i}+3\mathbf{j}+4\mathbf{k}$. Hence we can take $\mathbf{n}=\mathbf{v}=\mathbf{i}+3\mathbf{j}+4\mathbf{k}$. Therefore the equation for the plane is

$$\begin{aligned}\mathbf{n}\cdot\vec{P_0P} &= 0\Rightarrow(1)(x-2)+(3)(y-4)+(4)(z-5)=0\\ &\Rightarrow x+3y+4z=2+12+20=34\\ &\Rightarrow \boxed{x+3y+4z=34}.\end{aligned}$$

p.694, pr.25

(c) 9 points Find the area of the triangle whose vertices are $A(1,-1,1)$, $B(0,1,1)$, and $C(1,0,-1)$.

Solution: We use the formula $\dfrac{1}{2}(|\vec{AB}\times\vec{AC}|)$.

$$\begin{aligned}\vec{AB} &= (0-1)\mathbf{i}+(1-(-1))\mathbf{j}+(1-1)\mathbf{k}\\ &= -\mathbf{i}+2\mathbf{j}\\ \vec{AC} &= (1-1)\mathbf{i}+(0-(-1))\mathbf{j}+(-1-1)\mathbf{k}\\ &= \mathbf{j}-2\mathbf{k}\\ &\Rightarrow \vec{AB}\times\vec{AC}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 2 & 0\\ 0 & 1 & -2\end{vmatrix}\\ &= -4\mathbf{i}-2\mathbf{j}-\mathbf{k}\end{aligned}$$

Therefore, the required area is

$$\frac{1}{2}|\vec{AB}\times\vec{AC}|=\frac{1}{2}\sqrt{(-4)^2+(-2)^2+(-1)^2}=\frac{1}{2}\sqrt{21}$$

p.687, pr.47