Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives.**
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

Do not write in the table to the right.

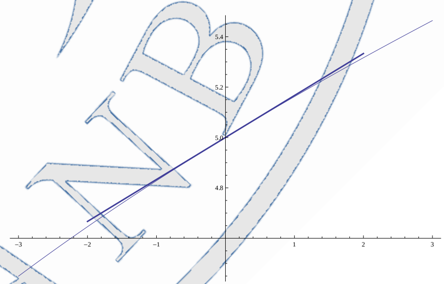
Problem	Points	Score
1	18	
2	20	
3	19	
4	20	
5	23	
Total:	100	

1. (a) 8 Points Let $f(x) = x^2 - 4x - 5$, $x > 2$. Find the value of df^{-1}/dx at the point $x = 0$.

Solution: Method I:Even though we can find a formula for f^{-1} in this case, we note that $y = 0$ corresponds $x = 5$, and since, $f'(x) = 2x - 4$To find the value of the derivative of f^{-1} at $x = 0$, we use the inverse function theorem (namely the Theorem 7.1).

$$(f^{-1})'(0) = \frac{1}{f'(5)} == \frac{1}{2(5)-4} = \boxed{\frac{1}{6}}$$

The figure on the right is the graph of f^{-1} with the tangent at the point $x = 0$.



Method II: We shall find the formula for f^{-1}. Let $y = x^2 - 4x - 5$. Then $y = (x^2 - 4x + 4) - 4 - 5 \Rightarrow y + 9 = (x-2)^2 \Rightarrow x - 2 = \pm\sqrt{y+9} \Rightarrow x = 2 \pm \sqrt{y+9}$. Therefore, since $x > 2$ is the given restriction on the domain for f^{-1}, the formula for f^{-1} is $f^{-1}(x) = 2 + \sqrt{x+9}$. Hence the derivative we want to find is $(f^{-1})'(x) = \dfrac{1}{2\sqrt{x+9}}$ and its value is

$$(f^{-1})'(0) = \frac{1}{2\sqrt{0+9}} = \frac{1}{2\sqrt{9}} = \frac{1}{6}.$$

p.368, pr.42

(b) 10 Points $y = \sqrt{\ln\sqrt{x}} \Rightarrow dy/dx = ?$

Solution: By applying the Chain Rule and logarithmic derivative rule, we have $y = \sqrt{\ln\sqrt{x}} = \left(\ln x^{1/2}\right)^{1/2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\ln x^{1/2}\right)^{-1/2} \cdot \frac{d}{dx}(\ln x^{1/2}) = \frac{1}{2}\left(\ln x^{1/2}\right)^{-1/2} \cdot \frac{1}{x^{1/2}}\frac{d}{dx}(x^{1/2})$. Hence we have $\frac{dy}{dx} = \frac{1}{2}\left(\ln x^{1/2}\right)^{-1/2} \cdot \frac{1}{x^{1/2}} \cdot \frac{1}{2}x^{-1/2} = \boxed{\frac{1}{4x\sqrt{\ln\sqrt{x}}}}$

Alternatively, much easier, we can use implicit differentiation.
$y = \sqrt{\ln\sqrt{x}} \Rightarrow y^2 = \ln\sqrt{x} = \frac{1}{2}\ln x.$
Now, the last thing is easy to differentiate.

$$\frac{d}{dx}y^2 = \frac{d}{dx}\left(\frac{1}{2}\ln x\right) \Rightarrow 2y\frac{dy}{dx} = \frac{1}{2}\frac{1}{x} \Rightarrow 2y\frac{dy}{dx} = \frac{1}{2}\frac{1}{x}\frac{1}{2y} = \frac{1}{4xy} \Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{4x\sqrt{\ln\sqrt{x}}}}$$

p.376, pr.30

2. (a) 10 Points $\lim_{x\to\infty}(\ln x)^{1/x} = ?$

Solution: Graphically, the figure on the right suggests that the line $y = 1$ is a horizontal asymptote. Let us verify this algebraically.
This limit leads to the indeterminate form ∞^0. Let $y = (\ln x)^{1/x}$. Then

$$\ln y = \ln(\ln x)^{1/x} = \frac{\ln(\ln x)}{x}.$$

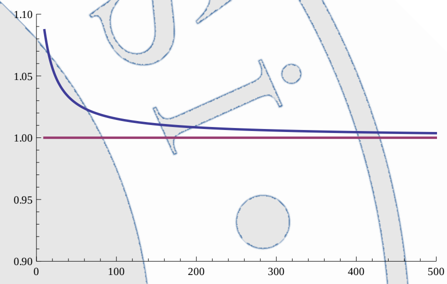
Now

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln\ln x}{x} = \lim_{x\to\infty}\frac{\left(\frac{1}{x\ln x}\right)}{1} = 0.$$

Therefore,

$$\lim_{x\to\infty}(\ln x)^{1/x} = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^0 = \boxed{1}.$$

p.402, pr.53

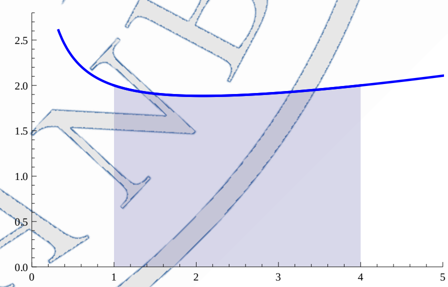


(b) 10 Points $\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx = ?$

Solution: Let $u = x^{1/2}$. Then $du = \frac{1}{2}x^{-1/2}\,dx$. So $2du = \frac{dx}{\sqrt{x}}$. When $x = 1$, we have $u = 1$ and when $x = 4$, we have $u = 2$. Therefore, we have

$$\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int_1^2 2^u\,du = 2\left[\frac{2^u}{\ln 2}\right]_1^2 = \frac{2}{\ln 2}(2^2 - 2^1) = \boxed{\frac{4}{\ln 2}}$$

p.385, pr.88



3. (a) 7 Points $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = ?$

Solution: $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = \cot\left(-\frac{\pi}{3}\right) = -\frac{1}{\sqrt{3}}$

p.413, pr.12

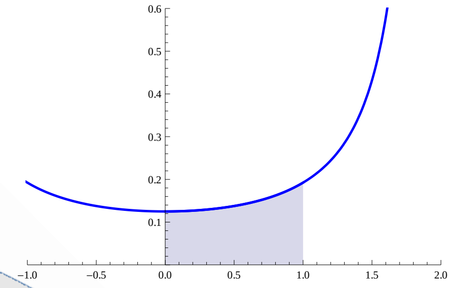
(b) 12 Points $\int_0^1 \frac{dx}{(4-x^2)^{3/2}} = ?$

Solution: Let $x = 2\sin\theta$ where $-\pi/2 \le \theta \le \pi/2$. Then $dx = 2\cos\theta\, d\theta$. The restriction $0 \le x \le 1$ implies the restriction $0 \le \theta \le \dfrac{\pi}{6}$, , $(4-x^2)^{3/2} = 8\cos^3\theta$;

$$\int_0^1 \frac{dx}{(4-x^2)^{3/2}} = \int_0^{\pi/6} \frac{2\cos\theta\, d\theta}{8\cos^3\theta}$$

$$= \frac{1}{4}\int_0^{\pi/6} \frac{d\theta}{\cos^2\theta} = \frac{1}{4}[\tan\theta]_0^{\pi/6} = \frac{\sqrt{3}}{12} = \boxed{\frac{1}{4\sqrt{3}}}$$

p.452, pr.24



4. (a) 10 Points $\displaystyle\int xe^{3x}\,dx = ?$

Solution: We will integrate by parts. Let $u = x$ and $dv = e^{3x}\,dx$. Then we have $du = dx$ and $v = \frac{1}{3}e^{3x}$. Integration by parts formula yields

$$\int xe^{3x}\,dx = \int u\,dv = uv - \int v\,du = x\left(\frac{1}{3}e^{3x}\right) - \int \frac{1}{3}e^{3x}\,dx = \boxed{\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C}$$

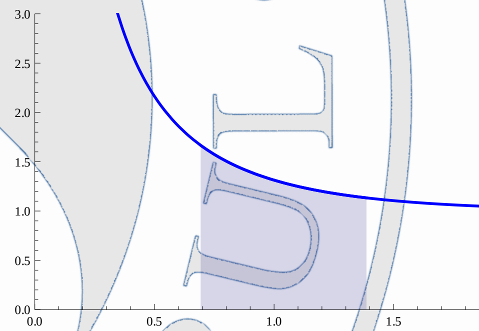
p.441, pr.8

(b) 10 Points $\displaystyle\int_{\ln 2}^{\ln 4} \coth x\,dx = ?$

Solution: The shaded area on the right has value that exactly equals the definite integral we are asked to find. to compute this, let $u = \sinh x$ and so $du = \cosh x\,dx$. When $x = \ln 2$, we have $u = \sinh(\ln 2) = \frac{1}{2}\left(e^{\ln 2} - e^{-\ln 2}\right) = \frac{1}{2}\left(2 - \frac{1}{2}\right) = \frac{3}{4}$ and when $x = \ln 4$, we have $u = \sinh(\ln 4) = \frac{1}{2}\left(e^{\ln 4} - e^{-\ln 4}\right) = \frac{1}{2}\left(4 - \frac{1}{4}\right) = \frac{15}{8}$. Now the integral becomes

$$\begin{aligned}
\int_{\ln 2}^{\ln 4} \coth x\,dx &= \int_{\ln 2}^{\ln 4} \frac{\cosh x}{\sinh x}\,dx = \int_{3/4}^{15/8} \frac{1}{u}\,du = [\ln|u|]_{3/4}^{15/8} \\
&= \ln\left(\frac{15}{8}\right) - \ln\left(\frac{3}{4}\right) \\
&= \ln(15) - \ln(8) - \ln(3) + \ln(4) \\
&= \cancel{\ln(3)} + \ln(5) - \ln(2) - \cancel{\ln(4)} - \cancel{\ln(3)} + \cancel{\ln(4)} \\
&= \boxed{\ln\frac{5}{2}}
\end{aligned}$$

p.452, pr.24



5. (a) 13 Points $\displaystyle\int_0^1 \frac{x^3}{x^2+2x+1}\,dx = ?$

Solution: By long division of polynomials, we have

$$\begin{array}{r|l}
 & x - 2 \\
x^2+2x+1 & x^3 + 0x^2 + 0x + 0 \\
 & -x^3 - 2x^2 \quad - x \\
\hline
 & -2x^2 \quad - x + 0 \\
 & 2x^2 + 4x + 2 \\
\hline
 & 3x + 2
\end{array}$$

Therefore,

$$\frac{x^3}{x^2+2x+1} = (x-2) + \frac{3x+2}{(x+1)^2}.$$

We decompose the integrand in the following way:

$$\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

Clearing the fractions changes to

$$3x+2 = A(x+1) + B \Rightarrow A = 3, A+B = 2 \Rightarrow A = 3, B = -1.$$

Thus,

$$\int_0^1 \frac{x^3}{x^2+2x+1}\,dx = \int_0^1 (x-2)\,dx + 3\int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{(x+1)^2} = \left[\frac{x^2}{2} + 2x + 3\ln|x+1| + \frac{1}{x+1}\right]_0^1$$

$$= \left(\frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2}\right) - (1) = \boxed{3\ln 2 - 2}$$

p.461, pr.17

(b) 10 Points Investigate the convergence/divergence for $\displaystyle\int_4^\infty \frac{2}{t^{3/2}-1}\,dt$.

Solution: First note that $f(t) := \dfrac{2}{t^{3/2}-1} > 0$ for all $t \geq 4$. We wish to use the Limit Comparison Test. Let $g(t) := \dfrac{1}{t^{3/2}} > 0$. Then we compute the limit

$$L := \lim_{t\to\infty} \frac{f(t)}{g(t)} = \lim_{t\to\infty} \frac{\frac{2}{t^{3/2}-1}}{\frac{1}{t^{3/2}}} = 2\lim_{t\to\infty} \frac{t^{3/2}}{t^{3/2}-1} = 2.$$

Now $0 < L = 2 < +\infty$ and $\displaystyle\int_4^\infty \frac{1}{t^{3/2}}\,dt = \lim_{b\to\infty}\int_4^b \frac{1}{t^{3/2}}\,dt = \lim_{b\to\infty}\left[\frac{t^{-3/2+1}}{-3/2+1}\right]_4^b = -2\lim_{b\to\infty}\left(\frac{1}{\sqrt{b}} - 2\right) = 4$ giving that $\displaystyle\int_4^\infty \frac{1}{t^{3/2}}\,dt$ converges. Thus by the Limit Comparison test, the given integral converges.

Alternatively, (Direct) Comparison Test could be more convenient to apply. Notice that when $t \geq 4$, we have

$$0 < \frac{2}{t^{3/2}-1} \leq \frac{2}{2t^{3/2}} = \frac{1}{t^{3/2}}$$

We have shown above that $\displaystyle\int_4^\infty \frac{1}{t^{3/2}}\,dt$ converges. Hence by (Direct) Comparison Test $\displaystyle\int_4^\infty \frac{2}{t^{3/2}-1}\,dt$ converges too.

p.487, pr.57