Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 60 min.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (a) 10 Points Find the value of $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$.

Solution: $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = \cot\left(-\frac{\pi}{3}\right) = \boxed{-\frac{1}{\sqrt{3}}}$

p.413, pr.12

(b) 15 Points Find the limit $\lim_{x\to 0}(e^x + x)^{1/x}$

Solution: As $x \to 0$, we have $e^x + x \to 1$ and $1/x \to \infty$ and so $(e^x+x)^{1/x} \to 1^\infty$. Thus this limit leads to the indeterminate 1^∞. Now let $y = (e^x+x)^{1/x}$. Then $\ln y = \ln(e^x+x)^{1/x} = \frac{\ln(e^x+x)}{x}$ has the indeterminate $\frac{0}{0}$ and so L'Hôpital's Rule applies. Now by L'Hôpital's Rule

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(e^x+x)}{x} = \lim_{x\to 0}\frac{\frac{d}{dx}\ln(e^x+x)}{\frac{d}{dx}x} = \lim_{x\to 0}\frac{\frac{e^x+1}{e^x+x}}{1} = \lim_{x\to 0}\frac{e^x+1}{e^x+x} = \frac{e^0+1}{e^0+0} = \frac{1+1}{1+0} = \boxed{2}$$

Now we have $\ln y \to 2$ as $x \to 0$ and so $y = e^{\ln y} \to e^2$. Therefore $\boxed{\lim_{x\to 0}(e^x+x)^{1/x} = e^2.}$

p.414, pr.96

2. (a) 12 Points Evaluate $\displaystyle\int \frac{dx}{\left(x^2-1\right)^{3/2}}, \quad x > 1.$

Solution: Let $x = \sec\theta$ with $0 < \theta < \pi/2$. Then $dx = \sec\theta\tan\theta\, d\theta$. Also we have $\left(x^2-1\right)^{3/2} = \left(\sec^2\theta - 1\right)^{3/2} = \left(\tan^2\theta\right)^{3/2} = \tan^3\theta$. Now the integral becomes

$$\begin{aligned}\int \frac{dx}{\left(x^2-1\right)^{3/2}} &= \int \frac{\sec\theta\tan\theta\, d\theta}{\tan^3\theta} \\ &= \int \frac{\cos\theta\, d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = \boxed{-\frac{x}{\sqrt{x^2-1}} + C}\end{aligned}$$

p.452, pr.25

(b) 13 Points $\displaystyle\int \frac{x+4}{x^2+5x-6}\, dx = ?$

Solution: The appropriate partial fraction decomposition takes the form

$$\frac{x+4}{x^2+5x-6} = \frac{x+4}{(x-1)(x+6)} = \frac{A}{x-1} + \frac{B}{x+6}$$

Then by clearing the fractions, we have the identity

$$x+4 = A(x+6) + B(x-1) \Rightarrow x+4 = x(A+B) + 6A - B \Rightarrow \begin{cases} A+B=1 \\ 6A-B=4 \end{cases} \Rightarrow \begin{cases} A = 5/7 \\ B = 2/7 \end{cases}$$

Therefore

$$\int \frac{x+4}{x^2+5x-6}\, dx = \int \frac{5/7}{x-1}\, dx + \int \frac{2/7}{x+6}\, dx = \boxed{\frac{5}{7}\ln|x-1| + \frac{2}{7}\ln|x+6| + C}$$

p.461, pr.11

3. (a) 12 Points $\displaystyle\int \frac{\ln x}{x^2}\,\mathrm{d}x = ?$

Solution: We shall integrate by parts. Let $u = \ln x$, $dv = \dfrac{1}{x^2}$. Then $du = \dfrac{1}{x}\,\mathrm{d}x$ and choose $v = -\dfrac{1}{x}$. Now substitute this into the formula $\displaystyle\int u\,dv = uv - \int v\,du$.

$$= -\frac{\ln x}{x} - \int -\frac{1}{x}\frac{1}{x}\,\mathrm{d}x = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,\mathrm{d}x$$

Now apply the power rule and add constant to get

$$\int \frac{\ln x}{x^2}\,\mathrm{d}x = \boxed{-\frac{\ln x}{x} - \frac{1}{x} + C}$$

p.442, pr.35

(b) 13 Points $\displaystyle\int_0^{\pi} \sin^5\left(\frac{x}{2}\right)\mathrm{d}x = ?$

Solution: Let $y = \dfrac{x}{2}$ and so $dy = \dfrac{1}{2}\,\mathrm{d}x$. Hence the indefinite integral is

$$\int \sin^5\left(\frac{x}{2}\right)\mathrm{d}x = 2\int \sin^5 y\,\mathrm{d}y = 2\int \sin^4 y \sin y\,\mathrm{d}y = 2\int \left(\sin^2 y\right)^2 \sin y\,\mathrm{d}y = 2\int \left(1-\cos^2 y\right)^2 \sin y\,\mathrm{d}y$$

Letting now $u = \cos y$ so that $du = -\sin y\,\mathrm{d}y$, we have

$$\begin{aligned}
\int \sin^5\left(\frac{x}{2}\right)\mathrm{d}x &= \int \left(1-\cos^2 y\right)^2 \sin y\,\mathrm{d}y = -\int \left(1-u^2\right)^2\mathrm{d}u = \int \left(-u^4+2u^2-1\right)\mathrm{d}u \\
&= \left[-\frac{1}{5}u^5 + \frac{2}{3}u^3 - u\right] + C = -\frac{1}{5}\cos^5 y + \frac{2}{3}\cos^3 y - \cos y + C \\
&= -\frac{1}{5}\cos^5\left(\frac{x}{2}\right) + \frac{2}{3}\cos^3\left(\frac{x}{2}\right) - \cos\left(\frac{x}{2}\right) + C
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\int_0^{\pi} \sin^5\left(\frac{x}{2}\right)\mathrm{d}x &= 2\left[-\frac{1}{5}\cos^5\left(\frac{x}{2}\right) + \frac{2}{3}\cos^3\left(\frac{x}{2}\right) - \cos\left(\frac{x}{2}\right)\right]_0^{\pi} \\
&= 2\left(-\frac{1}{5}\cos^5\left(\frac{\pi}{2}\right) + \frac{2}{3}\cos^3\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{2}\right)\right) - 2\left(-\frac{1}{5}\cos^5\left(\frac{0}{2}\right) + \frac{2}{3}\cos^3\left(\frac{0}{2}\right) - \cos\left(\frac{0}{2}\right)\right) \\
&= \frac{2}{5} - \frac{4}{3} + 2 = \boxed{\frac{16}{15}}
\end{aligned}$$

p.448, pr.8

4. (a) 11 Points Let $f(x)=\dfrac{\sqrt{x}}{\sqrt{x}-3}$. Find $f^{-1}(x)$. Justify your answer.

Solution: Let $y=\dfrac{\sqrt{x}}{\sqrt{x}-3}$. Then we must solve for x. Then we have

$$y=\frac{\sqrt{x}}{\sqrt{x}-3}\Rightarrow y(\sqrt{x}-3)=\sqrt{x}\Rightarrow y\sqrt{x}-3y=\sqrt{x}\Rightarrow y\sqrt{x}-\sqrt{x}=3y\Rightarrow \sqrt{x}(y-1)=3y\Rightarrow \sqrt{x}=\frac{3y}{y-1}$$

$$\Rightarrow x=\left(\frac{3y}{y-1}\right)^2$$

$$\Rightarrow y=\left(\frac{3x}{x-1}\right)^2=f^{-1}(x)$$

To check this is indeed the inverse we want, we verify the identities $f(f^{-1}(x))=x$ and $f^{-1}(f(x))=x$ for suitable domain members x. To this end, notice that

$$f(f^{-1}(x))=\frac{\sqrt{f^{-1}(x)}}{\sqrt{f^{-1}(x)}-3}=\frac{\sqrt{\left(\frac{3x}{x-1}\right)^2}}{\sqrt{\left(\frac{3x}{x-1}\right)^2}-3}=\frac{\frac{3x}{x-1}}{\frac{3x}{x-1}-3}=\frac{3x}{3x-3(x-1)}=\frac{3x}{3}=x$$

and

$$f^{-1}(f(x))=\left(\frac{3f(x)}{f(x)-1}\right)^2=\left(\frac{3\frac{\sqrt{x}}{\sqrt{x}-3}}{\frac{\sqrt{x}}{\sqrt{x}-3}-1}\right)^2=\frac{9x}{(\sqrt{x}-(\sqrt{x}-3))^2}=\frac{9x}{9}=x$$

p.368, pr.32

(b) 14 Points Let $y=\left(\dfrac{(t+1)(t-1)}{(t-2)(t+3)}\right)^5$. **Use logarithmic differentiation** to find $\left.\dfrac{dy}{dt}\right|_{t=5}$

Solution:

$$\ln y=\ln\left(\frac{(t+1)(t-1)}{(t-2)(t+3)}\right)^5=5\ln\left(\frac{(t+1)(t-1)}{(t-2)(t+3)}\right)=5\left(\ln[(t+1)(t-1)]-\ln[(t-2)(t+3)]\right)$$
$$=5\left(\ln(t+1)+\ln(t-1)-\ln(t-2)-\ln(t+3)\right)$$

Now we differentiate the last equation implicitly

$$\frac{1}{y}\frac{dy}{dt}=5\left(\frac{1}{t+1}+\frac{1}{t-1}-\frac{1}{t-2}-\frac{1}{t+3}\right)$$
$$\frac{dy}{dt}=5y\left(\frac{1}{t+1}+\frac{1}{t-1}-\frac{1}{t-2}-\frac{1}{t+3}\right)$$
$$\frac{dy}{dt}=5\left(\frac{(t+1)(t-1)}{(t-2)(t+3)}\right)^5\left(\frac{1}{t+1}+\frac{1}{t-1}-\frac{1}{t-2}-\frac{1}{t+3}\right)$$

We now evaluate the derivative at $t=5$.

$$\left.\frac{dy}{dt}\right|_{t=5}=5\left(\frac{(t+1)(t-1)}{(t-2)(t+3)}\right)^5\left.\left(\frac{1}{t+1}+\frac{1}{t-1}-\frac{1}{t-2}-\frac{1}{t+3}\right)\right|_{t=5}$$
$$=5(1)^5\left(\frac{1}{5+1}+\frac{1}{5-1}-\frac{1}{5-2}-\frac{1}{5+3}\right)=5\,\frac{4+6-8-3}{24}=\boxed{-\frac{5}{24}}$$

p.430, pr.27