Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- Calculators, cell phones off and away!.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

Do not write in the table to the right.

Problem	Points	Score
1	24	
2	25	
3	27	
4	24	
Total:	100	

1. (a) 13 Points Find the *value* of $\int_{-\pi/4}^{\pi/4} \cosh(\tan\theta)\sec^2\theta\, d\theta$.

Solution: Let $u = \tan\theta$ and so $du = \sec^2\theta\, d\theta$. When $\theta = -\pi/4$, we have $u = \tan(-\pi/4) = -1$ and when $\theta = \pi/4$, we have $u = \tan(\pi/4) = 1$. Hence

$$\begin{aligned}\int_{-\pi/4}^{\pi/4} \cosh(\tan\theta)\sec^2\theta\, d\theta &= \int_{-1}^{1} \cosh u\, du \\ &= [\sinh u]_{-1}^{1} \\ &= \sinh 1 - \sinh(-1) = \sinh 1 - (-\sinh(-1)) = \sinh 1 + \sinh 1 \\ &= 2\sinh 1 = \boxed{e + e^{-1}}\end{aligned}$$

p.317, pr.14(b)

(b) 11 Points *Evaluate* the integral $\int x\sec^2 x\, dx$.

Solution: We integrate by parts. Let $u = x$ and so $dv = \sec^2 x\, dx$. Then $du = dx$ and choose $v = \tan x$. Hence

$$\begin{aligned}\int x\sec^2 x\, dx &= \int u\, dv \\ &= uv - \int v\, du \\ &= x\tan x - \int \tan x\, dx \\ &= x\tan x - (-\ln|\cos x|) + c = \boxed{x\tan x + \ln|\cos x| + c}\end{aligned}$$

p.241, pr.65(a)

2. (a) [13 Points] *Evaluate* the integral $\int \sin^2(2\theta)\cos^3(2\theta)\,d\theta$.

Solution:

$$\begin{aligned}\int \sin^2(2\theta)\cos^3(2\theta)\,d\theta &= \int \sin^2(2\theta)\cos^2(2\theta)\cos(2\theta)\,d\theta \\ &= \int \sin^2(2\theta)\left(1-\sin^2(2\theta)\right)\cos(2\theta)\,d\theta\end{aligned}$$

Let $u = \sin(2\theta)$ and so $du = 2\cos(2\theta)\,d\theta$. Hence

$$\begin{aligned}\int \sin^2(2\theta)\cos^3(2\theta)\,d\theta &= \frac{1}{2}\int \sin^2(2\theta)\left(1-\sin^2(2\theta)\right)\cos(2\theta)\,d\theta \\ &= \frac{1}{2}\int u^2(1-u^2)\,du = \frac{1}{2}\int (u^2-u^4)\,du \\ &= \frac{1}{2}\left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right] + c \\ &= \frac{1}{6}\sin^3(2\theta) - \frac{1}{10}\sin^5(2\theta) + c\end{aligned}$$

p.345, pr.32(c)

(b) [12 Points] Suppose $a_n = (3^n + 5^n)^{1/n}$. Investigate the convergence of the *sequence* $\{a_n\}_{n=1}^{\infty}$. If it converges, *find its limit*.

Solution: We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(3^n+5^n)^{1/n} = \lim_{n\to\infty}\exp\left[\ln(3^n+5^n)^{1/n}\right] = \lim_{n\to\infty}\exp\left[\frac{\ln(3^n+5^n)}{n}\right] = \exp\lim_{n\to\infty}\left[\frac{\ln(3^n+5^n)}{n}\right]$$

$$= \exp\lim_{n\to\infty}\left[\frac{\dfrac{3^n\ln 3 + 5^n\ln 5}{3^n+5^n}}{1}\right]$$

$$= \exp\lim_{n\to\infty}\left[\frac{\frac{3^n}{5^n}\ln 3 + \ln 5}{\frac{3^n}{5^n}+1}\right] = \exp\lim_{n\to\infty}\left[\frac{\left(\frac{3}{5}\right)^n\ln 3 + \ln 5}{\left(\frac{3}{5}\right)^n + 1}\right] = \exp\ln 5 = \boxed{5}$$

Hence the sequence converges.

p.452, pr.24

3. (a) 15 Points Find the *value* of $\displaystyle\int_0^1 \frac{dx}{(4-x^2)^{3/2}}$.

Solution: Use the method of trig substitutions. So let $x = 2\sin\theta$ with $\theta \in [-\pi/2, \pi/2]$. Then $dx = 2\cos\theta\, d\theta$. When $x = 0$, we have $2\sin\theta = 0$ and so $\theta = 0$. When $x = 1$, we have $2\sin\theta = 1$ and so $\theta = \pi/6$. Since

$$(4-x^2)^{3/2} = (4-(2\sin\theta)^2)^{3/2} = (4-x^2)^{3/2} = (4-4\sin^2\theta)^{3/2} = 4^{3/2}(1-\sin^2\theta)^{3/2} = (2^2)^{3/2}(\cos^2\theta)^{3/2} = 8\cos^3\theta,$$

then the integral becomes

$$\begin{aligned}
\int_0^1 \frac{dx}{(4-x^2)^{3/2}} &= \int_0^{\pi/6} \frac{2\cos\theta\, d\theta}{8\cos^3\theta} = \frac{1}{4}\int_0^{\pi/6} \frac{\cancel{2\cos\theta}\, d\theta}{\cancel{2\cos\theta}\cos^2\theta} = \frac{1}{4}\int_0^{\pi/6} \frac{1}{\cos^2\theta}\, d\theta \\
&= \frac{1}{4}\int_0^{\pi/6} \sec^2\theta\, d\theta \\
&= \frac{1}{4}[\tan\theta]_0^{\pi/6} = \frac{1}{4}\tan\pi/6 - \frac{1}{4}\tan 0 = \frac{1}{4}\frac{1}{\sqrt{3}} \\
&= \boxed{\frac{1}{4\sqrt{3}}}
\end{aligned}$$

p.241, pr.65(a)

(b) 12 Points Find the *value* of $\displaystyle\int_2^\infty \frac{2\,dt}{t^2-1}$.

Solution: This is an improper integral of type I (having infinite limits of integration). The integrand $f(t) = \dfrac{2}{t^2-1}$ is continuous on $[2,\infty)$. Before we attempt to evaluate the given integral, we first use partial fraction decomposition for the integrand.

$$f(t) = \frac{2}{t^2-1} = \frac{2}{(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1} \Rightarrow A(t+1) + B(t-1) = 2$$

If $t = 1$, we have $2A = 2$ and so $A = 1$ and if $t = -1$, we have $-2B = 2$ which implies $B = -1$. Hence

$$\int \frac{2}{t^2-1}\, dt = \int \left(\frac{1}{t-1} + \frac{-1}{t+1}\right) dt = \ln|t-1| + \ln|t+1| = \ln\left|\frac{t-1}{t+1}\right| + c$$

So, by definition of integral of type I, we have

$$\begin{aligned}
\int_2^\infty \frac{2\,dt}{t^2-1} &= \lim_{b\to\infty}\int_2^b \frac{2\,dt}{t^2-1} \\
&= \lim_{b\to\infty}\left[\ln\left|\frac{t-1}{t+1}\right|\right]_2^b = \lim_{b\to\infty}\left(\ln\left|\frac{b-1}{b+1}\right| - \ln\left|\frac{2-1}{2+1}\right|\right) = \lim_{b\to\infty}\left(\ln\left|\frac{1-1/b}{1+1/b}\right| - \ln\left|\frac{1}{3}\right|\right) \\
&= \cancelto{0}{\ln 1} - \cancelto{0}{\ln 1} + \ln 3 = \boxed{\ln 3}
\end{aligned}$$

The given integral, therefore, *converges* and has *value* $\ln 3$.

p.236, pr.7

4. (a) 12 Points Determine if the series

$$\left(\frac{-2}{3}\right)^2+\left(\frac{-2}{3}\right)^3+\left(\frac{-2}{3}\right)^4+\left(\frac{-2}{3}\right)^5+\left(\frac{-2}{3}\right)^6+\cdots$$

converges or diverges. If it converges, find its *sum*. p.83, pr.45

Solution: If we delete the first two terms, namely, 1 and $-2/3$, from the geometric series with first term $a=1$ and ratio $r=-2/3$, we get the given series. Since $|r|=\frac{2}{3}<1$, the series converges and together with the first two terms its sum exactly equals $\frac{a}{1-r}=\frac{1}{1+2/3}=\frac{3}{5}$. So the sum of given series is equal to $\frac{3}{5}-1+\frac{2}{3}=\frac{3}{5}-\frac{1}{3}=\frac{4}{15}$. Alternatively, we calculate as follows.

$$\begin{aligned}\left(\frac{-2}{3}\right)^2+\left(\frac{-2}{3}\right)^3+\left(\frac{-2}{3}\right)^4+\left(\frac{-2}{3}\right)^5+\left(\frac{-2}{3}\right)^6+\cdots &= \left(\frac{-2}{3}\right)^2\left[1+\left(\frac{-2}{3}\right)+\left(\frac{-2}{3}\right)^2+\left(\frac{-2}{3}\right)^3+\left(\frac{-2}{3}\right)^6+\cdots\right]\\ &= \left(\frac{-2}{3}\right)^2\frac{1}{1-\left(-\frac{2}{3}\right)}\\ &= \frac{4}{9}\frac{3}{5}=\boxed{\frac{4}{15}}\end{aligned}$$

p.583, pr.32

(b) 12 Points *Investigate the convergence* of $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$.

Solution: We use the Limit Comparison Test. Let $a_n=\frac{\sqrt{n}}{n^2+1}>0$ and $b_n=\frac{\sqrt{n}}{n^2}=\frac{1}{n^{3/2}}>0$. Then

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\sqrt{n}}{n^2+1}\frac{n^{3/2}}{1}=\lim_{n\to\infty}\frac{n^2}{n^2+1}=\lim_{n\to\infty}\frac{1}{1+1/n^2}=\frac{1}{1+0}=1$$

Hence $0<1<\infty$. This shows that $\sum a_n$ and $\sum b_n$ both act the same. But $\sum b_n=\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ is a convergent p-series ($p=3/2>1$). Hence by Limit Comparison Test the given series converges.

p.452, pr.24