Math 114 Summer 2017 **First Exam** July 13, 2017

Your Name / Ad - Soyad (**75 min.**) Signature / İmza

Student ID # / Öğrenci No (**mavi tükenmez!**)

Problem	1	2	3	4	Total
Points:	22	23	25	30	100
Score:					

Time limit is **75 minutes**. If you need more room on your exam paper, you may use the empty spaces on the paper. You have to show your. Answers without reasonable-even if your results are true- work will either get zero or very little credit.

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1. (a) (11 Points) Use the trigonometric substitution $x = \sec\theta$ to evaluate $\int \frac{x}{\sqrt{x^2-1}}\,\mathrm{d}x$.

Solution: Since $x = \sec\theta$, $\mathrm{d}x = \sec\theta\tan\theta\ \mathrm{d}\theta$ and also $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$;

$$\begin{aligned}\int \frac{x}{\sqrt{x^2-1}}\,\mathrm{d}x &= \int \frac{\sec\theta}{\cancel{\tan\theta}}\sec\theta\cancel{\tan\theta}\ \mathrm{d}\theta \\ &= \int \sec^2\theta\ \mathrm{d}\theta \\ &= \tan\theta + C \\ &= \sqrt{\sec^2\theta - 1} + C \\ &= \boxed{\sqrt{x^2-1}+C}\end{aligned}$$

p.72, pr.15

(b) (11 Points) If $a_n = \frac{\ln(n+1)}{\sqrt{n}}$ does the sequence $\{a_n\}_{n=1}^{\infty}$ converge? If so, find its limit.

Solution:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\ln(n+1)}{\sqrt{n}} \overset{(\mathrm{L'H})}{=} \lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty}\frac{2\sqrt{n}}{n+1}$$

$$= \lim_{n\to\infty}\frac{\left(\frac{2}{\sqrt{n}}\right)}{1+\left(\frac{1}{n}\right)} = \frac{0}{1+0} = \boxed{0}$$

p.94, pr.34

2. (a) (10 Points) Find the derivative of $y = \ln(\cosh z)$.

Solution:

$$\begin{aligned}\frac{dy}{dz} &= \frac{d}{dz}(\ln(\cosh z)) = \frac{1}{\cosh z}\frac{d}{dz}(\cosh z)\\ &= \frac{\sinh z}{\cosh z}\\ &= \boxed{\tanh z}\end{aligned}$$

p.72, pr.8

(b) (13 Points) Evaluate the integral $\int \sin^2 x \cos^5 x \, \mathrm{d}x$.

Solution: First notice that if we let $y = \sin x$ and so $\mathrm{d}y = \cos x \, \mathrm{d}x$, then

$$\begin{aligned}\int \sin^2 x \cos^5 x \, \mathrm{d}x &= \int \sin^2 x \cos^4 x \cos x \, \mathrm{d}x = \int \sin^2 x (1-\sin^2 x)^2 \cos x \, \mathrm{d}x\\ &= \int y^2(1-y^2)^2 \, \mathrm{d}y\end{aligned}$$

Hence

$$\begin{aligned}\int \sin^2 x \cos^5 x \, \mathrm{d}x &= \int y^2(1-y^2)^2 \, \mathrm{d}y\\ &= \int y^2(1-2y^2+y^4) \, \mathrm{d}y = \int (y^2-2y^4+y^6) \, \mathrm{d}y\\ &= \frac{1}{3}y^3 - \frac{2}{5}y^5 + \frac{1}{7}y^7 + C\\ &= \boxed{\frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C}\end{aligned}$$

p.83, pr.52

3. (a) (11 Points) Evaluate the integral $\int \frac{\ln x}{x^2}\,\mathrm{d}x$.

Solution: We shall integrate by parts. Let $u = \ln x$ and so $dv = \frac{1}{x^2}\,dx$. Then $du = \frac{1}{x}\,\mathrm{d}x$ and choose $v = -\frac{1}{x}$. Therefore

$$\begin{aligned}
\int \frac{\ln x}{x^2}\,\mathrm{d}x &= \int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u \\
&= (\ln x)\left(-\frac{1}{x}\right) - \int \left(-\frac{1}{x}\right)\frac{1}{x}\,\mathrm{d}x \\
&= -\frac{\ln x}{x} + \int \frac{1}{x^2}\,\mathrm{d}x \\
&= -\frac{\ln x}{x} - \frac{1}{x} + C \\
&= \boxed{-\frac{\ln x}{x} - \frac{1}{x} + C}
\end{aligned}$$

p.95, pr.68

(b) (14 Points) Evaluate the integral $\int \frac{x^3+x^2}{x^2+x-2}\,\mathrm{d}x$.

Solution: By long division of polynomials, we have

$$\begin{array}{r|l}
 & x+0 \\
x^2+x-2 & x^3+x^2+0x+0 \\
 & -x^3-x^2+2x \\ \hline
 & 0x^2+2x+0 \\
 & 0x^2+0x+0 \\ \hline
 & 2x+0
\end{array}$$

Therefore,

$$\frac{x^3+x^2}{x^2+x-2} = x + \frac{2x}{x^2+x-2}.$$

We decompose the integrand in the following way:

$$\frac{2x}{x^2+x-2} = \frac{2x}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

Clearing the fractions changes to

$$2x = A(x-1) + B(x+2) \Rightarrow 2x = x(A+B) - A + 2B \Rightarrow A+B = 2, -A+2B = 0 \Rightarrow A = 4/3, B = 2/3.$$

Thus,

$$\begin{aligned}
\int \frac{x^3+x^2}{x^2+x-2}\,\mathrm{d}x &= \int \left(x + \frac{4/3}{x+2} + \frac{2/3}{x-1}\right)\mathrm{d}x = \int x\,\mathrm{d}x + \int \frac{4/3}{x+2}\,\mathrm{d}x + \int \frac{2/3}{x-1}\,\mathrm{d}x \\
&= \boxed{\frac{1}{2}x^2 + \frac{4}{3}\ln|x+2| + \frac{2}{3}\ln|x-1| + C}
\end{aligned}$$

p.112, pr.26

4. Aşağıdaki serlerin yakınsaklığını araştırınız. Investigate the convergence or divergence of the following series.

(a) (10 Points) $\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{(n+1)(n+2)}+\cdots$

◯ Converges. ◯ Diverges. Series' Sum: ______________

Solution: First, by partial fraction decomposition of the general term, we have

$$a_n=\frac{1}{(n+1)(n+2)}=\frac{1}{n+1}-\frac{1}{n+2}$$
$$s_n=a_1+a_2+a_3+\cdots+a_{n-1}+a_n$$
$$=\left(\frac{1}{2}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{5}}\right)+\cdots+\left(\cancel{\frac{1}{n}}-\cancel{\frac{1}{n+1}}\right)+\left(\cancel{\frac{1}{n+1}}-\frac{1}{n+2}\right)$$
$$=\frac{1}{2}-\frac{1}{n+2}$$
$$\Rightarrow \lim_{n\to\infty} s_n=\lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{n+2}\right)=\boxed{\frac{1}{2}}$$

p.72, pr.8

(b) (10 Points) $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$

◯ Converges. ◯ Diverges. Test Used: ______________

Solution: Let $a_n=\frac{\sqrt{n}}{n^2+1}>0$ for each $n\geq 1$. We have

$$n^2+1>n^2\Rightarrow n^2+1>n^{3/2}n^{3/2}=\sqrt{n}\ \cdot n^{3/2}\Rightarrow\frac{n^2+1}{\sqrt{n}}>\frac{\cancel{\sqrt{n}}\,n^{3/2}}{\cancel{\sqrt{n}}}$$
$$\Rightarrow\frac{\sqrt{n}}{n^2+1}<\frac{1}{n^{3/2}}\text{ for each }n\geq 1\ .$$

Now $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ is a convergent p-series with $p=3/2$. hence by the Direct Comparison Test the given series *converges*.

Also use the Limit comparison Test with $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$.

p.82, pr.35

(c) (10 Points) $\sum_{n=1}^{\infty}\frac{n!}{(2n+1)!}$

◯ Converges. ◯ Diverges. Test Used: ______________

Solution: Let $a_n=\frac{n!}{(2n+1)!}$. Use the Ratio Test. Then, since $a_{n+1}=\frac{(n+1)!}{(2(n+1)+1)!}=\frac{(n+1)!}{(2n+3)!}$

$$\frac{a_{n+1}}{a_n}=\frac{(n+1)!}{(2n+3)!}\frac{(2n+1)!}{n!}=\frac{(n+1)\cancel{n!}}{(2n+3)(2n+2)\cancel{(2n+1)!}}\frac{\cancel{(2n+1)!}}{\cancel{n!}}=\frac{1}{2(2n+3)}$$
$$\Rightarrow\rho=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{1}{2(2n+3)}=0$$

Since $\rho=0<1$, it follows by Ratio Test, the series *converges*.

p.82, pr.35