Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (a) 10 Points Determine if the series $\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$ converges or diverges. Name the test you use.

Solution: Let $a_n := \frac{(\ln n)^n}{n^n} > 0$ for each $n = 1, 2, 3, \cdots$. The Root Test yields

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{(\ln n)^n}{n^n}} = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = 0.$$

Since $\rho = 0 < 1$, the series converges by the Root Test.

p.368, pr.42

(b) 10 Points Does the series $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n\sqrt{n}}$ converge absolutely, conditionally, or diverge? Justify your answer.

Solution: Notice first that $\cos(n\pi) = (-1)^n$ for each $n = 1, 2, \cdots$. Therefore

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}.$$

This series converges absolutely, since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^{3/2}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

is a convergent p-series with $p = \frac{3}{2} > 1$.

p.376, pr.30

2. (a) 14 Points Find the interval of convergence of $\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n}$.

Solution: Let $u_n = \frac{(x+1)^{2n}}{9^n}$. Then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x+1)^{2n+2}}{9^{n+1}}\frac{9^n}{(x+1)^{2n}}\right| < 1 \Rightarrow \frac{(x+1)^2}{9}\lim_{n\to\infty}(1) < 1 \Rightarrow (x+1)^2 < 9 \Rightarrow |x+1| < 3$$

$$\Rightarrow -3 < x+1 < 3 \Rightarrow -4 < x < 2; \text{ when } x = -4 \text{ we have } \sum_{n=0}^{\infty}\frac{(-3)^{2n}}{9^n} = \sum_{n=0}^{\infty} 1 \text{ which diverges;}$$

$$\text{at } x = 2 \text{ we have } \sum_{n=0}^{\infty}\frac{(3)^{2n}}{9^n} = \sum_{n=0}^{\infty} 1 \text{ which also diverges; the interval of convergence is } -4 < x < 2; \text{ the series}$$

$$\sum_{n=0}^{\infty}\frac{(x+1)^{2n}}{9^n} = \sum_{n=0}^{\infty}\left(\left(\frac{x+1}{3}\right)^2\right)^n \text{ is a convergent geometric series}$$

p.402, pr.53

(b) 6 Points Find the sum of the series in part (a).

Solution: For $-4 < x < 2$ and $r = \frac{(x+1)^2}{9}$, the geometric series $\sum_{n=0}^{\infty} r^n$ has the sum

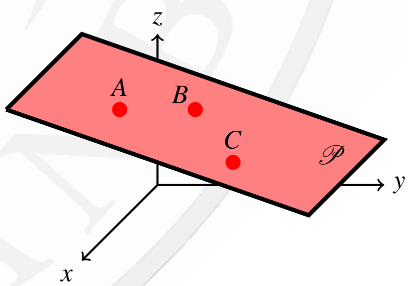
$$\frac{1}{1-r} = \frac{1}{1-\frac{(x+1)^2}{9}} = \frac{9}{9-(x+1)^2} = \frac{9}{9-x^2-2x-1} = \boxed{\frac{9}{8-x^2-2x}}$$

p.385, pr.88

3. (a) 10 Points Find the equation for the plane $\mathscr{P}$ through $A(1,1,-1)$, $B(2,0,2)$, and $C(0,-2,1)$.

Solution: First we find a normal vector to the plane:

$$\begin{aligned}
\vec{AB} &= (2-1)\mathbf{i} + (0-1)\mathbf{j} + (2-(-1))\mathbf{k} \\
&= \mathbf{i} - \mathbf{j} + 3\mathbf{k} \\
\vec{AC} &= (0-1)\mathbf{i} + (-2-1)\mathbf{j} + (1-(-1))\mathbf{k} \\
&= -\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} \\
&\Rightarrow \vec{AB}\times\vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix} \\
&= 7\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}
\end{aligned}$$



is normal to the plane

$$\Rightarrow 7(x-2) - 5(y-0) - 4(z-2) = 0$$

hence $\boxed{7x - 5y - 4z = 6}$ is the equation of the plane.

p.695, pr.23

(b) 10 Points Find the distance from the point $Q(3,-1,4)$ to the line $\mathscr{L}: \begin{cases} x = 4 - t, \\ y = 3 + 2t, \\ z = -5 + 3t \end{cases}$.

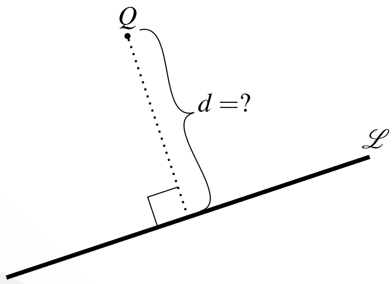
Solution: We shall use the distance formula $d = \dfrac{|\vec{PQ} \times \mathbf{v}|}{|\mathbf{v}|}$. Here $P(4,3,-5)$ is a point on $\mathscr{L}$ and $\mathbf{v} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ is a vector that is parallel to $\mathscr{L}$. Now we have $\vec{PQ} = -\mathbf{i} - 4\mathbf{j} + 9\mathbf{k}$ and so

$$\begin{aligned} \vec{PQ} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -4 & 9 \\ -1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} -4 & 9 \\ 2 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 9 \\ -1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & -4 \\ -1 & 2 \end{vmatrix} \mathbf{k} \\ &= -30\mathbf{i} - 6\mathbf{j} - 6\mathbf{k} \end{aligned}$$

Therefore, we have

$$d = \frac{|\vec{PQ} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{900+36+36}}{\sqrt{1+4+9}} = \frac{\sqrt{972}}{\sqrt{14}} = \frac{\sqrt{486}}{\sqrt{7}} = \boxed{\frac{3\sqrt{54}}{\sqrt{7}}}$$

p.695, pr.37



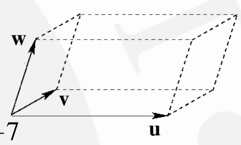
4. (a) [10 Points] Find the volume of the parallelepiped (box) determined by $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, and $\mathbf{w} = \mathbf{i} + 2\mathbf{k}$.

Solution: First, note that the Volume $= |(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w})|$. We have

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = (2) \begin{vmatrix} -1 & 1 \\ 0 & 2 \end{vmatrix} - (1) \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + (0) \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} = -7$$

Therefore, the volume of the parallelepiped is Volume $= |(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w})| = |-7| = \boxed{7}$.

p.441, pr.8



(b) [10 Points] Find parametric equations for the line $\mathscr{L}$ through $P(-2,0,3)$ and $Q(3,5,-2)$.

Solution: A vector parallel to $\mathscr{L}$ is

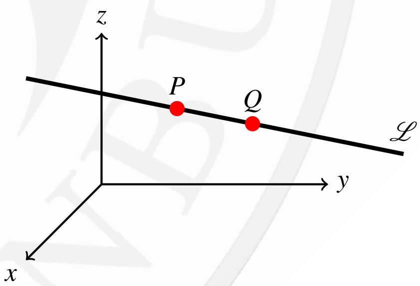
$$\mathbf{v} = (3-(-2))\mathbf{i} + (5-0)\mathbf{j} + (-2-3)\mathbf{k} = 5\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}$$

If we choose (x_0, y_0, z_0) as $(-2,0,3)$, we get the parametric equations

$$\boxed{\mathscr{L} : \begin{cases} x = -2 + 5t, \\ y = 0 + 5t, \\ z = 3 - 5t \end{cases} ,-\infty < t < +\infty}$$

Note that $t = 0$ determines the point $P(-2,0,3)$ whereas $t = 0$ determines $Q(3,5,-2)$. In fact, $0 \le t \le 1$ corresponds to the line segment joining these points.

p.452, pr.24



5. (a) [10 Points] Find a formula for the nth partial sum s_n and the sum of the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$.

Solution:

$$
\begin{aligned}
s_n &= a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n \\
&= \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) + \cdots + \left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}\right) + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) \\
&= \left(1 - \frac{1}{\sqrt{n+1}}\right)
\end{aligned}
$$

Hence the sum of the series is

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{\sqrt{n+1}}\right) = \boxed{1}.$$

p.413, pr.12

(b) 10 Points Find symmetric equations for the line in which the planes $3x - 6y - 2z = 3$ and $2x + y - 2z = 2$ intersect.

Solution: We begin by finding two points on the line. Any two points on the line would do, but we choose to find the points where the line pierces the yz-plane and the xz-plane. We get the former by setting $x = 0$ and solving the resulting equations $\begin{cases} -6y - 2z = 3 \\ y - 2z = 2 \end{cases}$ simultaneously. This yields the point $(0, -1/7, -15/14)$. Similarly, by setting $y = 0$, we get the equations $\begin{cases} 3x - 2z = 3 \\ 2x - 2z = 2 \end{cases}$. Solving these yields the point $(1, 0, 0)$. Consequently a vector parallel to the required line is

$$\mathbf{v} = (1-0)\mathbf{i} + (0 - (-1/7))\mathbf{j} + (0 - (-15/14))\mathbf{k} = \mathbf{i} + \frac{1}{7}\mathbf{j} + \frac{15}{14}\mathbf{k}$$

We can clear the denominators out by multiplying this vector by 15, we can take $\mathbf{v}$ to be $\mathbf{v} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$. Using $(1,0,0)$ for (x_0, y_0, z_0), we get

$$\boxed{\frac{x-1}{14} = \frac{y-0}{2} = \frac{z-0}{15}}$$

An alternative solution is based on the fact that the line of intersection of two planes is perpendicular to both of their normals. The vector $\mathbf{n_1} := 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ is normal to the first plane; $\mathbf{n_2} := 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is normal to the second. Since

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\mathbf{i} + \mathbf{j} + 15\mathbf{k},$$

the vector $\mathbf{v} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$ is parallel to the required line. Next, find any point on the line of intersection, for example, $(1,0,0)$, and proceed as in the earlier solution.

p.452, pr.24