Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 60 min.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (a) 13 Points Evaluate the *improper integral* $\displaystyle\int_{-8}^{1} \frac{dx}{x^{1/3}}$.

Solution: First notice that the integrand is discontinuous at $x = 0 \in [-8,1]$. Hence

$$\begin{aligned}
\int_{-8}^{1} \frac{dx}{x^{1/3}} &= \int_{-8}^{0} \frac{dx}{x^{1/3}} + \int_{0}^{1} \frac{dx}{x^{1/3}} = \lim_{b\to 0^-} \int_{-8}^{b} \frac{dx}{x^{1/3}} + \lim_{a\to 0^+} \int_{a}^{1} \frac{dx}{x^{1/3}} \\
&= \lim_{b\to 0^-} \left[\frac{x^{-1/3+1}}{-1/3+1}\right]_{-8}^{b} + \lim_{a\to 0^+} \left[\frac{x^{-1/3+1}}{-1/3+1}\right]_{a}^{1} \\
&= \frac{3}{2}\lim_{b\to 0^-} \left(b^{2/3} - (-8)^{2/3}\right) + \frac{3}{2}\lim_{a\to 0^+} \left(1^{2/3} - (a)^{2/3}\right) \\
&= \frac{3}{2}(0-(4)) + \frac{3}{2}(1-0) = \boxed{-\frac{9}{2} \text{ CONVERGES.}}
\end{aligned}$$

p.487, pr.6

(b) 12 Points Determine if the integral $\displaystyle\int_{\pi}^{\infty} \frac{1+\sin x}{x^2}\,dx$ converges or diverges. Explain your answer.

Solution: First note for any $x \in \mathbb{R}$ that $-1 \le \sin x \le 1$ and so $0 \le 1 + \sin x \le 2$. Hence for any $x \ge \pi$, we have

$$0 \le \frac{1+\sin x}{x^2} \le \frac{2}{x^2}.$$

Moreover the integral

$$\int_{\pi}^{\infty} \frac{2}{x^2}\,dx = \lim_{b\to\infty} \int_{\pi}^{b} \frac{2}{x^2}\,dx = \lim_{b\to\infty} \left[\frac{2x^{-2+1}}{-2+1}\right]_{\pi}^{b} = -2\lim_{b\to\infty}\left(\frac{1}{b} - \frac{1}{\pi}\right) = -2\left(0 - \frac{1}{\pi}\right) = \boxed{\frac{2}{\pi}}$$

converges. Therefore by Direct Comparison Test, the original integral CONVERGES.

p.487, pr.56

2. (a) $\boxed{\text{10 Points}}$ Find the limit of $\{a_n\}_{n=1}^{\infty}$ if $a_n = \left(2 - \frac{1}{2^n}\right)\left(3 + \frac{1}{2^n}\right)$.

Solution: First multiply and write this as

$$a_n = \left(2 - \frac{1}{2^n}\right)\left(3 + \frac{1}{2^n}\right) = 6 + \frac{2}{2^n} - \frac{3}{2^n} - \left(\frac{1}{2^n}\right)^2$$

Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(6 + \frac{2}{2^n} - \frac{3}{2^n} - \left(\frac{1}{2^n}\right)^2\right) = \lim_{n\to\infty}(6) + \lim_{n\to\infty}\left(\frac{2}{2^n}\right) - \lim_{n\to\infty}\left(\frac{3}{2^n}\right) - \left(\lim_{n\to\infty}\frac{1}{2^n}\right)^2 = \boxed{\text{6 CONVERGES.}}$$

p.542, pr.38

(b) $\boxed{\text{15 Points}}$ Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}} - \frac{(-1)^n}{5^n}\right)$.

Solution: This is the difference of two convergent geometric series $\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}$ and $\sum_{n=1}^{\infty}\frac{(-1)^n}{5^n}$. The first series is of the form $\sum_{n=1}^{\infty} ar^{n-1}$ where $a = 1$ and $r = 1/2$ and so has the sum $\frac{a}{1-r} = \frac{1}{1-1/2} = 2$. Similarly, the second series is of the form $\sum_{n=1}^{\infty} ar^{n-1}$ where $a = -1/5$ and $r = -1/5$ and so has the sum $\frac{a}{1-r} = \frac{-1/5}{1+1/5} = -1/6$. Finally the sum of given series is $\sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}} - \frac{(-1)^n}{5^n}\right) = 2 - (-1/6) = \boxed{\frac{13}{6}}$

p.551, pr.13

3. (a) $\boxed{\text{12 Points}}$ Determine if the series $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$ converges or diverges. Explain your answer.

Solution: Here we may use the Direct Comparison Test. For let $a_n = \dfrac{\ln n}{\sqrt{n}}$ and $b_n = \dfrac{1}{\sqrt{n}}$. Notice that if $n \geq 3$, then $a_n \geq b_n$. Since $\displaystyle\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series (with $p = 1/2 \leq 1$), we conclude that the given series diverges by Direct Comparison Test.

p.557, pr.20

(b) $\boxed{\text{13 Points}}$ Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+2)!}$ converges or diverges. Explain your answer.

Solution: Let $a_n = \dfrac{(n-1)!}{(n+2)!} > 0$ and $b_n = 1/n^3 > 0$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{(n-1)!}{(n+2)!}}{1/n^3} = \lim_{n\to\infty} \frac{\cancel{(n-1)!}}{(n+2)(n+1)n\cancel{(n-1)!}} \frac{n^3}{1} = \lim_{n\to\infty} \frac{n^2}{(n+2)(n+1)} = 1$$

So $0 < c = 1 < \infty$. Since $\displaystyle\sum_{n=1}^{\infty} 1/n^3$ is a convergent p-series ($p = 3 > 1$) by Limit Comparison Test, the given series CONVERGES.

p.562, pr.44

4. (a) $\boxed{\text{15 Points}}$ Find the *radius and interval of convergence* of $\displaystyle\sum_{n=1}^{\infty} \frac{(3x+1)^{n+1}}{2n+2}$.

Solution: Let $u_n = \dfrac{(3x+1)^{n+1}}{2n+2}$. Then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{\frac{(3x+1)^{n+2}}{2n+4}}{\frac{(3x+1)^{n+1}}{2n+2}}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|(3x+1)\frac{2n+2}{2n+4}\right| < 1 \Rightarrow |3x+1|\underbrace{\lim_{n\to\infty}\left(\frac{2+2/n}{2+4/n}\right)}_{=1} < 1$$

$$\begin{aligned} &\Rightarrow |3x+1| < 1 \\ &\Rightarrow -1 < 3x+1 < 1 \\ &\Rightarrow -\frac{2}{3} < x < 0 \end{aligned}$$

When $x = -\dfrac{2}{3}$, we have $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+2}$, a conditionally convergent series.

When $x = 0$, we have $\displaystyle\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{2n+2}$, a divergent series. So the radius of convergence is $R = 1/3$; the interval of convergence is

p.583, pr.32

(b) $\boxed{\text{10 Points}}$ Find the first four nonzero terms of the Maclaurin series for $f(x) = \ln(1+x)$.

Solution: $f(x) = \ln(1+x)$. We have

$f(x) = \ln(1+x)$	$f(0) = \ln(1+0) = \ln 1 = 0$
$f'(x) = \dfrac{1}{1+x}$	$f'(0) = \dfrac{1}{1+0} = 1$
$f''(x) = -\dfrac{1}{(1+x)^2}$	$f''(0) = -\dfrac{1}{(1+x)^2} = -1$
$f'''(x) = \dfrac{2}{(1+x)^3}$	$f'''(0) = \dfrac{2}{(1+0)^2} = 2$

Now from the definition of Maclauren series, we have

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = \boxed{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots}$$

p.588, pr.4