Your Name / Adınız - Soyadınız

Your Signature / İmza

Student ID # / Öğrenci No

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- **Calculators, cell phones off and away!**
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 70 min.

Do not write in the table to the right.

Problem	Points	Score
1	15	
2	24	
3	30	
4	31	
Total:	100	

1. [15 Points] Find the *radius and interval of convergence* of $\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n 2^n}$.

Solution: Let $u_n = \frac{(-1)^n (x+2)^n}{n 2^n}$. Then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(x+2)^{n+1}}{(n+1)2^{n+1}}}{\frac{(-1)^n(x+2)^n}{n2^n}}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x+2)}{2}(-1)\frac{n}{n+1}\right| < 1 \Rightarrow \frac{|x+2|}{2}\underbrace{\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)}_{=1} < 1$$

$$\Rightarrow |x+2| < 2$$
$$\Rightarrow -2 < x+2 < 2$$
$$\Rightarrow -4 < x < 0$$

When $x = -4$, we have $\sum_{n=1}^{\infty} \frac{1}{n}$, a divergent series.

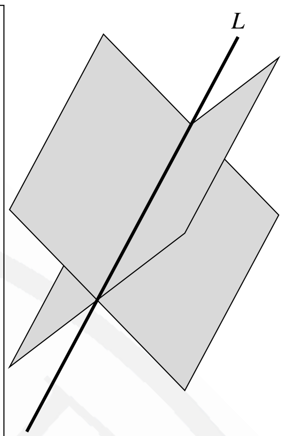
When $x = 0$, we have $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, the alternating harmonic series which converges conditionally.

The radius of convergence is $\boxed{R = 2}$.

The interval of convergence is $\boxed{-4 < x \leq 0}$ or is $(-4, 0]$.

p.626, pr.27

2. (a) 14 Points Find *parametric equations for the line* in which the planes $3x-6y-2z=3$ and $2x+y-2z=2$ intersect.



Solution: We begin by finding two points on the line. Any two on the line would do, but we choose to find the points where line pierces yz-plane and the xz-plane. We get the former by setting $x=0$ and solving the resulting equations $\begin{cases}-6y-2z=3\\ y-2z=2\end{cases}$ simultaneously. This yields the point $(0,-1/7,-15/14)$. Similarly, by setting $y=0$, we get the equations $\begin{cases}3x-2z=3\\ 2x-2z=2\end{cases}$. Solving these yields the point $(1,0,0)$. Consequently a vector parallel to the required line is

$$\mathbf{v}=(1-0)\mathbf{i}+(0-(-1/7))\mathbf{j}+(0+15/14)\mathbf{k}=\mathbf{i}+\frac{1}{7}\mathbf{j}+\frac{15}{14}\mathbf{k}$$

We can clear the denominators out by multiplying this vector by 14, we can take $14\mathbf{v}$ to be $\mathbf{v}=14\mathbf{i}+2\mathbf{j}+15\mathbf{k}$. Using $(1,0,0)$ for (x_0,y_0,z_0), we get

$$\boxed{\mathscr{L}:\begin{cases}x=1+14t,\\ y=0+2t,\\ z=0+15t\end{cases}}$$

An alternative solution is based on the fact that line of intersection for planes is perpendicular to both of their normals. The vector $\mathbf{n_1}:=3\mathbf{i}-6\mathbf{j}-2\mathbf{k}$ is normal to the first plane; $\mathbf{n_2}:=2\mathbf{i}+\mathbf{j}-2\mathbf{k}$ is normal to the second. Since

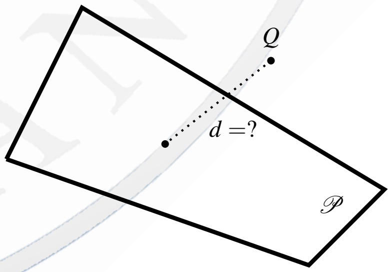
$$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & -6 & -2\\ 2 & 1 & -2\end{vmatrix}=14\mathbf{i}+2\mathbf{j}+15\mathbf{k},$$

the vector $\mathbf{v}=14\mathbf{i}+2\mathbf{j}+15\mathbf{k}$ is parallel to the required line. Next, find any point on the line of intersection, for example, $(1,0,0)$, and proceed as in the earlier solution.

$$\boxed{\mathscr{L}:\begin{cases}x=1+14t,\\ y=2t,\\ z=15t\end{cases}}$$

p.695, pr.23

(b) 10 Points Find the *distance* from the point $Q(1,1,1)$ to the plane $2x+y-2z=2$.



Solution: We shall use the distance formula $d=\left|\frac{\vec{PQ}\cdot\mathbf{n}}{|\mathbf{n}|}\right|$. Here $P(1,0,0)$ is a point on plane, say $\mathscr{P}$, and $\mathbf{n}=2\mathbf{i}+\mathbf{j}-2\mathbf{k}$ is a vector that is normal to $\mathscr{P}$. Now we have $\vec{PQ}=0\mathbf{i}+\mathbf{j}+\mathbf{k}$ and so

$$\vec{PQ}\cdot\mathbf{n}=(\mathbf{j}+\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}-2\mathbf{k})$$

Therefore, we have

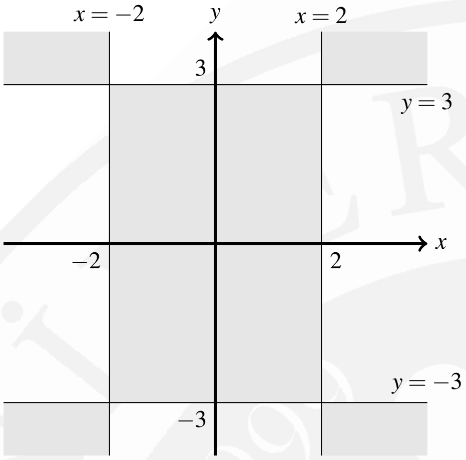
$$d=\left|\frac{\vec{PQ}\cdot\mathbf{n}}{|\mathbf{n}|}\right|=\left|\frac{(0)(2)+(1)(1)+(1)(-2)}{\sqrt{4+1+4}}\right|=\left|\frac{-1}{3}\right|=\boxed{\frac{1}{3}}$$

p.695, pr.37

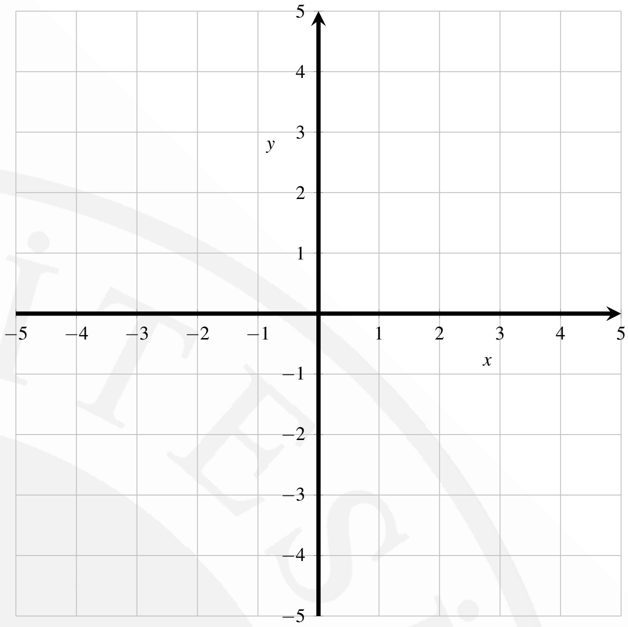
3. (a) 10 Points Find and sketch the *domain*, on the given grid, for $f(x,y) = \sqrt{(x^2-4)(y^2-9)}$.

Solution: Let $D \subset \mathbb{R}^2$ denote the domain of f. Suppose $(x,y) \in D$. Then $(x-2)(x+2)(y-3)(y+3) \geq 0$ must be satisfied. Hence

$$D = \{(x,y) \in \mathbb{R}^2 \mid (x-2)(x+2)(y-3)(y+3) \geq 0\}.$$



p.583, pr.32



(b) 10 Points Does the limit $\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$ exist? Why? Explain your answer.

Solution: The substitution $y = kx$ yields

$$\lim_{\substack{(x,y)\to(0,0)\\ k\neq -1}} \frac{x-y}{x+y} = \lim_{\substack{(x,kx)\to(0,0)\\ k\neq 0}} \frac{x-kx}{x+kx} = \lim_{x\to 0} \frac{\cancel{x}(1-k)}{\cancel{x}(1+k)} = \lim_{x\to 0} \frac{1-k}{1+k}$$
$$= \frac{1-k}{1+k}$$

This is the limit as $(x,y) \to (0,0)$ along the straight line of slope $k \neq -1$. Different values of k (such as 1 and 0) give different values for the limit. Hence if $(x,y) \to (0,0)$, along the line $y = x$ (where $k = 1$) the limit is $\frac{1-1}{1+1} = 0$, whereas if $(x,y) \to (0,0)$ along the line $y = 0$ (where $k = 0$) the limit is $\frac{1-0}{(1+0)} = 1$. Therefore *the given limit does not exist*.

p.811, pr.45

(c) 10 Points If $f(x,y) = x\ln(xy)$, find the *second derivative* f_{xy}.

Solution: First, notice that $f(x,y) = x\ln(xy) = x(\ln x + \ln y) = x\ln x + x\ln y$. Then

$$f_y = \frac{\partial f}{\partial y}(x\ln x + x\ln y) = \underbrace{\frac{\partial}{\partial y}(x\ln x)}_{=0} + x\underbrace{\frac{\partial}{\partial y}(\ln y)}_{1/y} = \frac{x}{y}$$

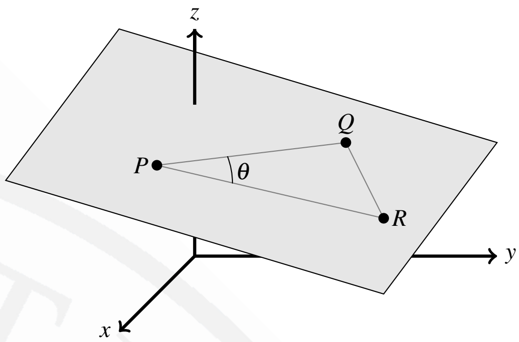
Differentiating with respect to y first, we have

$$f_{xy} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y}$$

p.821, pr.55 (f)

4. Let $\triangle(PQR)$ be the triangle having the vertices $P(-1,1,-1)$, $Q(1,1,1)$, and $R(1,2,3)$. Let M be the plane containing $\triangle(PQR)$.

   (a) 14 Points Find an *equation for the plane* M.



Solution: First we find a normal vector to the plane:

$$\begin{aligned}\vec{PQ} &= (1-(-1))\mathbf{i}+(1-1)\mathbf{j}+(1-(-1))\mathbf{k}\\ &= 2\mathbf{i}+2\mathbf{k}\\ \vec{PR} &= (1-(-1))\mathbf{i}+(2-1)\mathbf{j}+(3-(-1))\mathbf{k}\\ &= 2\mathbf{i}+\mathbf{j}+4\mathbf{k}\\ \Rightarrow \vec{PQ}\times\vec{PR} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 0 & 2\\ 2 & 1 & 4\end{vmatrix}\\ &= -2\mathbf{i}-4\mathbf{j}+2\mathbf{k}\end{aligned}$$

is normal to the plane

$$\Rightarrow -2(x+1)-4(y-1)+2(z+1)=0$$

hence $\boxed{x+2y-z=2}$ is the equation of the plane.

p.695, pr.23

   (b) 8 Points Find the *area* $\triangle(PQR)$.

Solution: We have

$$\begin{aligned}A(\triangle(PQR)) &= \frac{1}{2}|\vec{PQ}\times\vec{PR}|\\ &= \frac{1}{2}|-2\mathbf{i}-4\mathbf{j}+2\mathbf{k}| = \frac{1}{2}\sqrt{(-2)^2+(-4)^2+(2)^2} = \frac{1}{2}\sqrt{24} = \frac{2}{2}\sqrt{6} = \boxed{\sqrt{6}}\end{aligned}$$

p.241, pr.65(a)

   (c) 9 Points If θ is the *angle* at the vertex P, find $\cos\theta$ in $\triangle(PQR)$.

Solution:

$$\begin{aligned}\cos\theta = \frac{\vec{PQ}\cdot\vec{PR}}{|\vec{PQ}||\vec{PR}|} = \frac{(2\mathbf{i}+2\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}+4\mathbf{k})}{|2\mathbf{i}+2\mathbf{k}||2\mathbf{i}+\mathbf{j}+4\mathbf{k}|} &= \frac{(2)(2)+(0)(1)+(2)(4)}{\sqrt{2^2+0^2+2^2}\sqrt{2^2+1^2+4^2}}\\ &= \frac{12}{\sqrt{8}\sqrt{21}}\end{aligned}$$

p.241, pr.65(a)