Solutions to Exercise Sheet 1 are now on my website.

In the exams, you will typically not be told if an equation is linear, separable, exact, homogeneous, etc – you should be able to determine this yourself. You can use Exercises 15 and 16 to practise.

Exercise 15 (First Order ODEs). Find the general solutions of the following ODEs:

(a) $9yy' + 4x = 0$.

(b) $y' + (x+1)y^3 = 0$.

(c) $\frac{dx}{dt} = 3t(x+1)$.

(d) $y' + \csc y = 0$.

(e) $x' \sin 2t = x \cos 2t$.

(f) $y' = (y-1)\cot x$.

(g) $\frac{dy}{dx} + \left(\frac{2x+1}{x}\right) y = e^{-2x}$.

(h) $(3x^2 + y^2)dx - 2xydy = 0$.

(i) $y' = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$.

(j) $e^{\frac{x}{y}}(y-x)\frac{dy}{dx} + y(1+e^{\frac{x}{y}}) = 0$.

(k) $(2x+3y)dx + (3x+2y)dy = 0$.

(l) $(x^3 + \frac{y}{x})dx + (y^2 + \ln x)dy = 0$.

(m) $(e^x \sin y + \tan y)dx + (e^x \cos y + x\sec^2 y)dy = 0$.

(n) $ydx + (2x - ye^y)dy = 0$.

(o) $xy' + y = y^{-2}$

(p) $y' = y(xy^3 - 1)$.

(q) $(1+x^2)y' = 2xy(y^3 - 1)$.

Solution 15. Thanks to Prof. Eldem for these solutions.

(a) This is a separable equation. Thus, we have

$$\begin{aligned} 9y\,dy &= -4x\,dx \Longrightarrow \int 9y\,dy = -\int 4x\,dx + C \Longrightarrow \\ \frac{9}{2}y^2 &= -2x^2 + C \Longrightarrow y = \pm\sqrt{\frac{2}{9}C - \frac{4}{9}x^2} = \pm\frac{2}{3}\sqrt{C_1 - x^2}. \quad \left(C_1 = \frac{C}{2}\right). \end{aligned}$$

(b) This equation can be written as follows.

$$\begin{aligned} \frac{dy}{y^3} &= -(x+1)\,dx \Longrightarrow \int \frac{dy}{y^3} = -\int (x+1)\,dx + C \Longrightarrow \\ \frac{1}{2y^2} &= \frac{x^2}{2} + x + C \Longrightarrow y = \pm\sqrt{\frac{1}{x^2+2x+2C}}. \end{aligned}$$

(c) This separable equation can be written as follows.

$$\begin{aligned} \frac{dx}{(x+1)} &= 3t\,dt \Longrightarrow \int \frac{dx}{(x+1)} = \int 3t\,dt + C \Longrightarrow \\ \ln|(x+1)| &= \frac{3}{2}t^2 + C \Longrightarrow x(t) = C_1 e^{\frac{3}{2}t^2} - 1. \quad \left(C_1 = e^C\right). \end{aligned}$$

(d) This separable equation can be solved as follows.

$$\begin{aligned} \frac{dy}{dx} &= -\frac{1}{\sin y} \Longrightarrow -\sin y\,dy = dx \Longrightarrow -\int \sin y\,dy = \int dx + C \Longrightarrow \\ \cos y &= x + C \Longrightarrow y = \arccos(x+C). \end{aligned}$$

(e) This is a separable equation. Therefore, we get

$$\begin{aligned} \frac{dx}{x} &= \cot 2t\,dt \Longrightarrow \ln x = \int \frac{\cos 2t}{\sin 2t}\,dt + C = \frac{1}{2}\ln(\sin 2t) + C \Longrightarrow \\ x &= C_1\sqrt{\sin 2t}. \quad \left(C_1 = e^C\right). \end{aligned}$$

(f) Note that this is a separable equation which can be written as follows.

$$\begin{aligned} \frac{dy}{y-1} &= \cot x\,dx \Longrightarrow \int \frac{dy}{y-1} = \int \cot x\,dx + C \Longrightarrow \\ \ln(y-1) &= \ln(\sin x) + C \Longrightarrow y = 1 + C_1 \sin x. \quad \left(C_1 = e^C\right). \end{aligned}$$

(g) The integrating factor is

$$e^{\int\left(\frac{2x+1}{x}\right)dx} = xe^{2x}.$$

Consequently, we get

$$\begin{aligned} \frac{d}{dx}\left(yxe^{2x}\right) &= xe^{2x}e^{-2x} = x \Longrightarrow yxe^{2x} = \int x\,dx = \frac{x^2}{2} + C \Longrightarrow \\ y &= \frac{x}{2}e^{-2x} + \frac{C}{x}e^{-2x} = \left(\frac{x}{2} + \frac{C}{x}\right)e^{-2x} = \frac{x^2 + C_1}{2xe^{2x}}. \quad (C_1 = 2C). \end{aligned}$$

(h) Let $M(x,y) = 3x^2 + y^2$ and $N(x,y) = 2xy$. Then, we have
$$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x},$$
which implies that the equation is exact. Thus, it follows that
$$F(x,y) = \int (3x^2 + y^2)\, dx + g(y) = x^3 + xy^2 + g(y).$$
Taking the derivative with respect to y, we obtain
$$\begin{aligned} \frac{\partial F}{\partial y} &= 2xy + g'(y) = N(x,y) = 2xy \Longrightarrow g'(y) = 0 \Longrightarrow \\ g(y) &= C \Longrightarrow F(x,y) = x^3 + xy^2 = C_1, \quad (C_1 = -C). \end{aligned}$$

(i) This is a homogeneous equation and we let $v = y/x \Longrightarrow y = vx$. Then, we get
$$\begin{aligned} \frac{dy}{dx} &= v + x\frac{dv}{dx} \Longrightarrow v + x\frac{dv}{dx} = v + \tan(v) \Longrightarrow \frac{dv}{dx} = \frac{\tan(v)}{x} \Longrightarrow \int \frac{dv}{\tan(v)} = \int \frac{dx}{x} + C \Longrightarrow \\ \ln(\sin v) &= \ln x + C \Longrightarrow \sin v = C_1 x \Longrightarrow v = \arcsin(C_1 x) \Longrightarrow y = x \arcsin(C_1 x), \quad \left(C_1 = e^C\right). \end{aligned}$$

(j) **Solution 1:** Let $v = x/y \Longrightarrow y = x/v$. This implies that
$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{v} - \frac{x}{v^2}\frac{dv}{dx} \Longrightarrow \frac{1}{v} - \frac{x}{v^2}\frac{dv}{dx} = -\frac{(1+e^v)}{e^v(1-v)} \Longrightarrow \\ \frac{dv}{dx} &= \frac{v^2}{x}\left(\frac{(1+e^v)}{e^v(1-v)} + \frac{1}{v}\right) = \left(\frac{v^2(1+e^v)}{xe^v(1-v)} + \frac{v}{x}\right) \Longrightarrow \\ \frac{e^v(1-v)}{v(v+e^v)}\, dv &= \frac{dx}{x} \Longrightarrow \frac{dv}{v} - \frac{1+e^v}{v+e^v}\, dv = \frac{dx}{x} \Longrightarrow \\ \int \frac{dv}{v} - \int \frac{1+e^v}{v+e^v}\, dv &= \int \frac{dx}{x} + C \Longrightarrow \ln\left(\frac{v}{v+e^v}\right) = \ln x + C \Longrightarrow \\ \frac{v}{v+e^v} &= C_1 x \Longrightarrow \frac{1}{x + ye^{\frac{x}{y}}} = C_1, \quad \left(C_1 = e^C\right) \Longrightarrow \\ x + ye^{\frac{x}{y}} &= C_2. \end{aligned}$$

Solution 2:
$e^{\frac{x}{y}}(y-x)\frac{dy}{dx} + y(1+e^{\frac{x}{y}}) = 0 \Longrightarrow e^{\frac{x}{y}}(y-x) + y(1+e^{\frac{x}{y}})\frac{dx}{dy} = 0$. Then we use the substitution $v = x/y \Longrightarrow x = vy$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$.Then, we get

$$\begin{aligned} e^v(y - vy) + y(1+e^v)(v + y\frac{dv}{dy}) &= 0 \\ [e^v(1-v) + v(1+e^v)]dy + (1+e^v)ydv &= 0 \\ (e^v + v)dy &= -(1+e^v)ydv \\ \frac{dy}{y} &= -\frac{(1+e^v)}{e^v+v}dv \\ \int \frac{dy}{y} &= -\int \frac{(1+e^v)}{e^v+v}dv + C \\ \ln y &= -\ln(e^v + v) + C \\ y(e^v + v) &= C_1, \left(C_1 = e^C\right) \\ ye^{\frac{x}{y}} + x &= C_1 \end{aligned}$$

(k) Let $M(x,y) = 2x + 3y$ and $N(x,y) = 3x + 2y$. Then, we have
$$\frac{\partial M}{\partial y} = 3 = \frac{\partial N}{\partial x},$$
which implies that the equation is exact. Thus, it follows that
$$F(x,y) = \int (2x + 3y)\, dx + g(y) = x^2 + 3xy + g(y).$$
Taking the derivative with respect to y, we obtain
$$\begin{aligned} \frac{\partial F}{\partial y} &= 3x + g'(y) = N(x,y) = 3x + 2y \Longrightarrow g'(y) = 2y \Longrightarrow \\ g(y) &= y^2 + C \Longrightarrow F(x,y) = x^2 + 3xy + y^2 = C_1, \quad (C_1 = -C). \end{aligned}$$

(l) Let $M(x,y) = (x^3 + \frac{y}{x})$ and $N(x,y) = (y^2 + \ln x)$. Then, we have
$$\frac{\partial M}{\partial y} = \frac{1}{x} = \frac{\partial N}{\partial x},$$
which implies that the equation is exact. Thus, it follows that
$$F(x,y) = \int (x^3 + \frac{y}{x})\, dx + g(y) = \frac{x^4}{4} + y\ln x + g(y).$$
Taking the derivative with respect to y, we obtain
$$\begin{aligned} \frac{\partial F}{\partial y} &= \ln x + g'(y) = N(x,y) = y^2 + \ln x \Longrightarrow g'(y) = y^2 \Longrightarrow \\ g(y) &= \frac{y^3}{3} + C \Longrightarrow F(x,y) = \frac{x^4}{4} + y\ln x + \frac{y^3}{3} = C_1, \quad (C_1 = -C). \end{aligned}$$

(m) Let $M(x,y) = (e^x \sin y + \tan y)$ and $N(x,y) = (e^x \cos y + x\sec^2 y)$. Then, we have
$$\frac{\partial M}{\partial y} = e^x \cos y + \sec^2 y = \frac{\partial N}{\partial x},$$
which implies that the equation is exact. Thus, it follows that
$$F(x,y) = \int (e^x \sin y + \tan y)\, dx + g(y) = e^x \sin y + x\tan y + g(y).$$
Taking the derivative with respect to y, we obtain
$$\begin{aligned} \frac{\partial F}{\partial y} &= e^x \cos y + x\sec^2 y + g'(y) = N(x,y) = e^x \cos y + x\sec^2 y \Longrightarrow g'(y) = 0 \Longrightarrow \\ g(y) &= C \Longrightarrow F(x,y) = e^x \sin y + x\tan y = C_1, \quad (C_1 = -C). \end{aligned}$$

(n) Let $M(x,y) = y$ and $N(x,y) = (2x - ye^y)$. Then, we have

$$\frac{\partial M}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = 2$$

Then, we check

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{-1}{y}.$$

Consequently, y is an integrating factor. Thus, we get

$$M_1(x,y) = y^2 \quad \text{and} \quad N_1(x,y) = (2xy - y^2e^y)$$

which implies that $M_1(x,y)dx + N_1(x,y)dy = 0$ is exact. Thus, it follows that

$$F(x,y) = \int y^2\, dx + g(y) = y^2x + g(y).$$

Taking the derivative with respect to y, we obtain

$$\begin{aligned}\frac{\partial F}{\partial y} &= 2xy + g'(y) = N_1(x,y) = (2xy - y^2e^y) \Longrightarrow g'(y) = -y^2e^y \Longrightarrow \\ g(y) &= -y^2e^y + 2ye^y - 2e^y + C \Longrightarrow F(x,y) = y^2x - e^y(y^2 - 2y + 2) = C_1, \quad (C_1 = -C).\end{aligned}$$

(o) This equation can be written as follows.

$$y' + \frac{1}{x}y = \frac{1}{x}y^{-2}.$$

Hence, we have a Bernoulli equation with $n = -2$. Let $v = y^3 \Longrightarrow v' = 3y^2y'$. Thus, we have

$$3y^2y' + 3y^2\frac{1}{x}y = 3y^2\frac{1}{x}y^{-2} \Longrightarrow v' + 3\frac{v}{x} = \frac{3}{x}.$$

The integrating factor is x^3 and we get

$$\frac{d}{dx}\left(x^3v\right) = 3x^2 \Longrightarrow x^3v = x^3 + C \Longrightarrow v = 1 + \frac{C}{x^3} \Longrightarrow y = \frac{(x^3+C)^{1/3}}{x}.$$

(p) This equation can be written as follows.

$$y' + y = xy^4.$$

Hence, we have a Bernoulli equation with $n = 4$. Let $v = y^{-3} \Longrightarrow v' = -3y^{-4}y'$. Thus, we have

$$-3y^{-4}y' - 3y^{-4}y = -3x \Longrightarrow v' - 3v = -3x.$$

The integrating factor is e^{-3x} and we get

$$\begin{aligned}\frac{d}{dx}\left(e^{-3x}v\right) &= -3xe^{-3x} \Longrightarrow e^{-3x}v = xe^{-3x} + \frac{1}{3}e^{-3x} + C \Longrightarrow v = \frac{3Ce^{3x} + 3x + 1}{3} \\ &\Longrightarrow \quad y = \left(\frac{3}{3Ce^{3x} + 3x + 1}\right)^{\frac{1}{3}}.\end{aligned}$$

(q) This equation can be written as follows.

$$y' + \frac{2xy}{(1+x^2)} = \frac{2xy^4}{(1+x^2)}.$$

Hence, we have a Bernoulli equation with $n = 4$. Let $v = y^{-3} \Longrightarrow v' = -3y^{-4}y'$. Thus, s solutions. we have

$$-3y^{-4}y' - \frac{6xy^{-3}}{(1+x^2)} = -\frac{6x}{(1+x^2)} \Longrightarrow v' - \frac{6x}{(1+x^2)}v = -\frac{6x}{(1+x^2)}.$$

The integrating factor is $(1+x^2)^{-3}$ and we get

$$\begin{aligned}\frac{d}{dx}\left((1+x^2)^{-3}v\right) &= -6x(1+x^2)^{-4} \Longrightarrow (1+x^2)^{-3}v = (1+x^2)^{-3} + C \Longrightarrow v = 1 + C(1+x^2)^3 \\ &\Longrightarrow \quad y = \left(\frac{1}{1 + C(1+x^2)^3}\right)^{\frac{1}{3}}.\end{aligned}$$

Exercise 16 (Initial Value Problems). Solve the following IVPs:

(a) $\begin{cases} y' = x^3e^{-y} \\ y(2) = 0 \end{cases}$

(b) $\begin{cases} y\frac{dy}{dx} = 4x(y^2+1)^{\frac{1}{2}} \\ y(0) = 1 \end{cases}$

(c) $\begin{cases} y' = y\cot x \\ y(\frac{\pi}{2}) = 2 \end{cases}$

(d) $\begin{cases} y' + 3(y-1) = 2x \\ y(0) = 1 \end{cases}$

(e) $\begin{cases} \frac{dy}{dx} = \frac{10}{(x+y)e^{x+y}} - 1 \\ y(0) = 0 \end{cases}$

(f) $\begin{cases} (4x^2 - 2y^2)y' = 2xy \\ y(3) = -5 \end{cases}$

(g) $\begin{cases} (x-y)dx + (3x+y)dy = 0 \\ y(3) = -2 \end{cases}$

(h) $\begin{cases} \frac{dy}{dx} = \frac{x^3 - xy^2}{x^2y} \\ y(1) = 1 \end{cases}$

(i) $\begin{cases} (xy+1)ydx + (2y-)dy = 0 \\ y(0) = 3 \end{cases}$.

(j) $\begin{cases} y' - \frac{1}{x}y = y^2 \\ y(1) = 2 \end{cases}$

Solution 16. Thanks to Prof. Eldem for these solutions.

(a) This equation can be written as follows.

$$\frac{dy}{dx} = x^3e^{-y} \Longrightarrow e^y\, dy = x^3\, dx \Longrightarrow e^y = \frac{x^4}{4} + C \Longrightarrow y = \ln\left(\frac{x^4}{4} + C\right).$$

Since $y(2) = 0$, we get

$$0 = y(2) = \ln\left(\frac{2^4}{4} + C\right) \Longrightarrow C = -3 \Longrightarrow y = \ln\left(\frac{x^4}{4} - 3\right)$$

(b) This equation can be written as follows.

$$\frac{dy}{dx} = \frac{4x(y^2+1)^{\frac{1}{2}}}{y} \Longrightarrow \frac{y}{(y^2+1)^{\frac{1}{2}}}\,dy = 4x\,dx \Longrightarrow (y^2+1)^{\frac{1}{2}} = 2x^2 + C \Longrightarrow y = \sqrt{(2x^2+C)^2 - 1}.$$

Since $y(0) = 1$, we get

$$1 = y(0) = y = \sqrt{\left(2\,(0)^2 + C\right)^2 - 1} \Longrightarrow C = \sqrt{2} \Longrightarrow y = \sqrt{\left(2x^2 + \sqrt{2}\right)^2 - 1}$$

(c) This equation can be expressed as follows.

$$\frac{dy}{dx} = y\cot x \Longrightarrow \frac{dy}{y} = \cot x\,dx \Longrightarrow \ln y = \ln(\sin x) + C \Longrightarrow y = C_1 \sin x, \qquad \left(C_1 = e^C\right).$$

Since $y(\frac{\pi}{2}) = 2$, we get $2 = y(\frac{\pi}{2}) = C_1 \sin(\frac{\pi}{2}) \Longrightarrow C_1 = 2 \Longrightarrow y = 2\sin x$.

(d) This equation can be expressed as follows.

$$\begin{aligned}
\frac{dy}{dx} + 3y &= 2x + 3 \Longrightarrow e^{3x}\frac{dy}{dx} + 3ye^{3x} = (2x+3)\,e^{3x} \Longrightarrow \frac{d}{dx}\left(ye^{3x}\right) = (2x+3)\,e^{3x} \Longrightarrow ye^{3x} = \int (2x+3)\,e^{3x}\,dx \Longrightarrow \\
ye^{3x} &= \frac{2}{3}xe^{3x} - \frac{2}{3}\int e^{3x}\,dx + e^{3x} + C \Longrightarrow ye^{3x} = \frac{2}{3}xe^{3x} + \frac{7}{9}e^{3x} + C \Longrightarrow \\
y &= \frac{1}{9}\,(6x+7) + Ce^{-3x}.
\end{aligned}$$

Since $y(0) = 1$, we get $1 = y(0) = \frac{1}{9}\,(6(0)+7) + Ce^{-3(0)} \Longrightarrow C = 2/9 \Longrightarrow y = \frac{1}{9}\left(6x + 2e^{-3x} + 7\right)$.

(e) Let $x + y = v \Longrightarrow y = v - x$. Then, we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dv}{dx} - 1 = \frac{10}{ve^v} - 1 \Longrightarrow \frac{dv}{dx} = \frac{10}{ve^v} \Longrightarrow \int ve^v\,dv = \int 10\,dx + C \Longrightarrow \\
ve^v - \int e^v\,dv &= 10x + C \Longrightarrow ve^v - e^v = 10x + C \Longrightarrow (x+y-1)e^{x+y} = 10x + C.
\end{aligned}$$

Since $y(0) = 0 \Longrightarrow C = -1$. Thus, we get

$$(x+y-1)e^{x+y} = 10x - 1.$$

(f) Dividing both sides by x^2, we get $\left(4 - 2\left(\frac{y}{x}\right)^2\right)\frac{dy}{dx} = 2\,\frac{y}{x}$. Let $v = y/x \Longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$. Then, we have

$$\begin{aligned}
v + x\frac{dv}{dx} &= \frac{2v}{(4-2v^2)} \Longrightarrow x\frac{dv}{dx} = \frac{v}{(2-v^2)} - v = \frac{v^3 - v}{(2-v^2)} \Longrightarrow \\
\frac{dv}{dx} &= \frac{1}{x}\frac{v^3 - v}{(2-v^2)} \Longrightarrow \int \frac{(2-v^2)}{v^3 - v}\,dv = \int \frac{dx}{x} + C.
\end{aligned}$$

If we use partial fraction expansion for the first integral, we get

$$\frac{(2-v^2)}{v^3 - v} = \frac{A}{v} + \frac{B}{v-1} + \frac{D}{v+1}.$$

where $A = -2$, $B = 1/2$ and $D = 1/2$. This implies that

$$\begin{aligned}
\int \frac{(2-v^2)}{v^3 - v}\,dv &= \int \left(-\frac{2}{v} + \frac{1/2}{v-1} + \frac{1/2}{v+1}\right) = \ln x + C \Longrightarrow \\
\ln\left(\frac{(v^2-1)^{\frac{1}{2}}}{v^2}\right) &= \ln x + C \Longrightarrow \frac{\sqrt{v^2-1}}{v^2} = C_1 x \Longrightarrow \\
\frac{\sqrt{y^2 - x^2}}{y^2} &= C_1, \ \left(C_1 = e^C\right).
\end{aligned}$$

Since $y(3) = -5 \Longrightarrow \sqrt{\frac{25-9}{25}} = C_1 \Longrightarrow C_1 = \frac{4}{5}$. Consequently, we get

$$\frac{\sqrt{y^2 - x^2}}{y^2} = \frac{4}{5} \Longrightarrow y^2 - \frac{16}{25}y^4 + x^2 = 0.$$

(g) This equation can be written as follows.

$$\frac{dy}{dx} = -\frac{(x-y)}{(3x+y)} = -\frac{(1 - \frac{y}{x})}{(3 + \frac{y}{x})}.$$

Let $v = y/x \Longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$. Then, we get

$$\begin{aligned}
v + x\frac{dv}{dx} &= -\frac{(1-v)}{(3+v)} \Longrightarrow \frac{dv}{dx} = -\frac{1}{x}\left(\frac{(1-v)}{(3+v)} + v\right) = -\frac{1}{x}\left(\frac{(v^2+2v+1)}{(3+v)}\right) \Longrightarrow \\
\frac{(3+v)\,dv}{(v+1)^2} &= -\frac{dx}{x} \Longrightarrow \int \frac{A\,dv}{(v+1)} + \int \frac{B\,dv}{(v+1)^2} = -\ln x + C,
\end{aligned}$$

where $B = 2$ and $A = 1$. Consequently, we have

$$\int \frac{dv}{(v+1)} + \int \frac{2\,dv}{(v+1)^2} = -\ln x + C \Longrightarrow \ln(v+1) - \frac{2}{(v+1)} = -\ln x + C.$$

Substituting $v = y/x$, we get

$$\ln\left(\frac{y+x}{x}\right) - \frac{2x}{(y+x)} = -\ln x + C \Longrightarrow \ln(y+x) - \frac{2x}{(y+x)} = C.$$

Since $y(3) = -2$, it follows that

$$\ln(-2+3) - \frac{6}{(-2+3)} = C \Longrightarrow C = -6.$$

Consequently, we get

$$\ln(y+x) - \frac{2x}{(y+x)} + 6 = 0.$$

(h) **Solution 1:** This equation can be rearranged as follows.

$$\frac{dy}{dx} = \frac{x^3 - xy^2}{x^2 y} = \frac{1 - \left(\frac{y}{x}\right)^2}{\frac{y}{x}}.$$

Let $v = y/x \Longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$. Then, we get

$$\begin{aligned} v + x\frac{dv}{dx} &= \frac{1-v^2}{v} \Longrightarrow \frac{dv}{dx} = \frac{1}{x}\left(\frac{1-v^2}{v} - v\right) = \frac{1}{x}\left(\frac{1-2v^2}{v}\right) \Longrightarrow \\ \frac{v\,dv}{(1-2v^2)} &= \frac{dx}{x} \Longrightarrow \int \frac{v\,dv}{(1-2v^2)} = \ln x + C \Longrightarrow -\frac{1}{4}\ln\left|1-2v^2\right| = \ln x + C \Longrightarrow \\ \frac{1}{|(1-2v^2)|^{1/4}} &= e^C x \Longrightarrow \left|\left(1-2v^2\right)\right| = \frac{1}{e^{4C}x^4}. \end{aligned}$$

Since $y(1) = 1$, we get $v(1) = 1$ which implies that $C = 0$. Consequently, we get

$$\left|\left(1 - 2\left(\frac{y}{x}\right)^2\right)\right| = \frac{1}{x^4} \Longrightarrow \left|\left(x^2 - 2y^2\right)\right| = \frac{1}{x^2}.$$

Solution 2 : It is a exact equation also. $\frac{dy}{dx} = \frac{x^3 - xy^2}{x^2 y} \Longrightarrow (x^3 - xy^2)\,dx - x^2 y dy = 0.$

Let $M = x^3 - xy^2$ and $N = -x^2 y$.Then

$$\frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x}$$

Therefore

$$\begin{aligned} F(x,y) &= \int (x^3 - xy^2)dx + g(y) = \frac{x^4}{4} - \frac{x^2y^2}{2} + g(y) \Longrightarrow \\ \frac{\partial F}{\partial y} &= -x^2 y + g'(y) = -x^2 y \Longrightarrow g'(y) = 0 \\ g(y) &= C \Longrightarrow F(x,y) = \frac{x^4}{4} - \frac{x^2y^2}{2} + C = 0. \end{aligned}$$

Since $y(1) = 1$, we get $C = -\frac{1}{4} \Longrightarrow x^4 - 2x^2y^2 = 1.$

(i) Let $M = xy^2 + y$ and $N = 2y - x$. Then, we have

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{2xy + 1 - (-1)}{xy^2 + y} = \frac{2}{y}.$$

This implies that the integrating factor is $p(y) = y^{-2}$. Let $M_1 = x + y^{-1}$ and $N_1 = 2y^{-1} - xy^{-2}$.Then, we have

$$\frac{\partial M_1}{\partial y} = -\frac{1}{y^2} = \frac{\partial N_1}{\partial x}$$

which implies that the equation is exact. Thus, we get

$$\begin{aligned} F(x,y) &= \int \left(x + y^{-1}\right)\,dx + g(y) = \frac{x^2}{2} + \frac{x}{y} + g(y) \Longrightarrow \\ \frac{\partial F}{\partial y} &= -\frac{x}{y^2} + g'(y) = 2y^{-1} - xy^{-2} \Longrightarrow g'(y) = 2y^{-1} \Longrightarrow \\ g(y) &= 2\ln y + C \Longrightarrow F(x,y) = \frac{x^2}{2} + \frac{x}{y} + 2\ln y + C = 0. \end{aligned}$$

Since $y(0) = 3$, we get $C = -2\ln 3$. Therefore, it follows that

$$F(x,y) = \frac{x^2}{2} + \frac{x}{y} + 2\ln y = 2\ln 3.$$

(j) This is a Bernoulli equation with $n = 2$. Let $v = y^{1-2} = y^{-1}$. Then, it follows that

$$\frac{dv}{dx} = -y^{-2}\frac{dy}{dx} \Longrightarrow -y^{-2}y' + \frac{1}{x}y^{-1} = -1 \Longrightarrow \frac{dv}{dx} + \frac{v}{x} = -1.$$

Note that the integrating factor is $e^{\int \frac{dx}{x}} = x$. Thus we get

$$\begin{aligned} x\frac{dv}{dx} + v &= -x \Longrightarrow \frac{d}{dx}(xv) = -x \Longrightarrow xv = -\frac{x^2}{2} + C \Longrightarrow v = \frac{C}{x} - \frac{x}{2} \\ &\Longrightarrow y = \frac{2x}{2C - x^2}. \end{aligned}$$

Since $y(1) = 2$, we get $C = 1$. Consequently, we have

$$y = \frac{2x}{2 - x^2}.$$

Exercise 17 (Homogeneous Second Order Linear ODEs with constant coefficients). Solve the following IVPs:

(a) $\begin{cases} y'' - 3y' + 2y = 0 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$ (b) $\begin{cases} y'' + 4y' + 3y = 0 \\ y(0) = 2 \\ y'(0) = -1 \end{cases}$ (c) $\begin{cases} y'' + 3y' = 0 \\ y(0) = -2 \\ y'(0) = 3 \end{cases}$ (d) $\begin{cases} y'' + 5y' + 3y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$

Solution 17.

(a) The characteristic equation is $0 = r^2 - 3r + 2 = (r-1)(r-2)$. The roots are $r_1 = 1$ and $r_2 = 2$. Therefore the general solution to the ODE is $y = c_1 e^t + c_2 e^{2t}$ for constants c_1 and c_2.

The first initial condition gives $1 = y(0) = c_1 + c_2$. Since $y'(x) = c_1 e^t + 2c_2 e^{2t}$, the second initial condition gives $1 = y'(0) = c_1 + 2c^2$. It follows that $c_1 = 1$ and $c_2 = 0$.

Therefore the solution to the IVP is $y(t) = e^t$.

(b) $y = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}$

(c) $y = -1 - e^{-3t}$

(d) $y = \dfrac{13 + 5\sqrt{13}}{26} e^{\frac{(-5+\sqrt{13})t}{2}} + \dfrac{13 - 5\sqrt{13}}{26} e^{\frac{(-5-\sqrt{13})t}{2}}$

Exercise 18 (Fundamental Sets of Solutions)**.** In each of the following: Verify that y_1 and y_2 are solutions of the given ODE; calculate the Wronskian of y_1 and y_2; and determine if they form a fundamental set of solutions.

(a) $t^2 y'' - 2y = 0; \quad y_1(t) = t^2, \quad y_2(t) = t^{-1}$

(b) $y'' + 4y = 0; \quad y_1(t) = \cos 2t, \quad y_2(t) = \sin 2t$

(c) $y'' - 2y + y = 0; \quad y_1(t) = e^t, \quad y_2(t) = te^t$

(d) $(1 - x\cot x)y'' - xy' + y = 0 \ (0 < x < \pi); \quad y_1(x) = x, \quad y_2(x) = \sin x$

Solution 18.

(a) Clearly $t^2 y_1'' - 2y_1 = t^2(t^2)'' - 2t^2 = t^2(2) - 2t^2 = 0$ and $t^2 y_2'' - 2y_2 = t^2(t^{-1})'' - 2t^{-1} = t^2(2t^{-3} - 2t^{-1} = 0$.

Next we calculate that

$$W(y_1, y_2)(t) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \begin{pmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{pmatrix} = -1 + 2 = 1.$$

Since $W \neq 0$, y_1 and y_2 form a fundamental set of solutions of the ODE.

(b) Yes

(c) Yes

(d) Yes