2018–19 **MATH216 Mathematics IV – Solutions to Exercise Sheet 4** N. Course

Exercise 19 (Homogeneous Second Order Linear ODEs with constant coefficients). Find the general solution of the following ODEs:

(a) $y'' - 2y' + 2y = 0$

(b) $y'' + 2y' + 2y = 0$

(c) $y'' + 2y' - 8y = 0$

(d) $y'' - 2y' + 6y = 0$

(e) $y'' + 6y' + 13y = 0$

(f) $9y'' + 16y = 0$

(g) $y'' - 2y' + y = 0$

(h) $9y'' + 6y' + y = 0$

(i) $4y'' + 12y' + 9y = 0$

(j) $4y'' - 4y' - 3y = 0$

(k) $y'' - 2y' + 10y = 0$

(l) $y'' - 6y' + 9y = 0$

(m) $4y'' + 17y' + 4y = 0$

(n) $4y'' + 20y' + 25y = 0$

(o) $25y'' - 20y' + 4y = 0$

(p) $2y'' + 2y' + y = 0$

Solve the following IVPs:

(q) $\begin{cases} 9y'' + 6y' + 82y = 0 \\ y(0) = -1 \\ y'(0) = 2 \end{cases}$

(r) $\begin{cases} y'' - 6y' + 9y = 0 \\ y(0) = 0 \\ y'(0) = 2 \end{cases}$

Solution 19.

(a) The characteristic equation is
$$r^2 - 2r + 2 = 0.$$
Thus
$$r = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$$
Hence we have complex roots with $\lambda = 1$ and $\mu = 1$. The general solution to the ODE is therefore
$$y = c_1 e^t \cos t + c_2 e^t \sin t.$$

(b) $y = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$

(c) $y = c_1 e^{2t} + c_2 e^{-4t}$

(d) $y = c_1 e^t \cos \sqrt{5}t + c_2 e^t \sin \sqrt{5}t$

(e) $y = c_1 e^{-3t} \cos 2t + c_2 e^{-3t} \sin 2t$

(f) $y = c_1 \cos \frac{4}{3}t + c_2 \sin \frac{4}{3}t$

(g) $y = c_1 e^t + c_2 t e^t$

(h) $y = c_1 e^{-\frac{t}{3}} + c_2 t e^{-\frac{t}{3}}$

(i) $y = c_1 e^{-\frac{3t}{2}} + c_2 t e^{-\frac{3t}{2}}$

(j) $y = c_1 e^{-\frac{t}{2}} + c_2 e^{\frac{3t}{2}}$

(k) $y = c_1 e^t \cos 3t + c_2 e^t \sin 3t$

(l) $y = c_1 e^{3t} + c_2 t e^{3t}$

(m) $y = c_1 e^{-\frac{t}{4}} + c_2 e^{-4t}$

(n) $y = c_1 e^{-\frac{5t}{2}} + c_2 t e^{-\frac{5t}{2}}$

(o) $y = c_1 e^{\frac{2t}{5}} + c_2 t e^{\frac{2t}{5}}$

(p) $y = c_1 e^{-\frac{t}{2}} \cos \frac{t}{2} + c_2 e^{-\frac{t}{2}} \sin \frac{t}{2}$

(q) $y = -e^{-\frac{t}{3}} \cos 3t + \frac{5}{9} e^{-\frac{t}{3}} \sin 3t$

(r) The characteristic equation is
$$0 = r^2 - 6r + 9 = (r-3)^2$$
which implies that we have the repeated root $r = 3$. Therefore the general solution to the ODE is
$$y = c_1 e^{3t} + c_2 t e^{3t}.$$
Since
$$y' = 3c_1 e^{3t} + c_2 e^{3t} + 3c_2 t e^{3t}$$
we have that
$$0 = y(0) = c_1 + 0$$
$$2 = y'(0) = 3c_1 + c_2 + 0,$$
which imples that $c_1 = 0$ and $c_2 = 2$. Therefore the solution to the IVP is
$$y = 2te^{3t}.$$

Exercise 20 (Reduction of Order). In each of the following problems:

(i) Check that y_1 solves the ODE;

(ii) Use the method of reduction of order to find a second, linearly independent solution, y_2 [HINT: Start with $y_2(t) = v(t)y_1(t)$.];

(iii) Check that your y_2 solves the ODE; and

(iv) Calculate the Wronskian of y_1 and y_2.

(a) $t^2y'' + 2ty' - 2y = 0, \quad t > 0; \quad y_1(t) = t$

(b) $t^2y'' - 4ty' + 6y = 0, \quad t > 0; \quad y_1(t) = t^2$

(c) $t^2y'' + 3ty' + y = 0, \quad t > 0; \quad y_1(t) = t^{-1}$

(d) $t^2y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0; \quad y_1(t) = t$

(e) $xy'' - y' + 4x^3y = 0, \quad x > 0; \quad y_1(x) = \sin x^2$

(f) $(x-1)y'' - xy' + y = 0, \quad x > 1; \quad y_1(x) = e^x$

Solution 20.

(a) (i) First we calculate that $y_1' = 1$, $y_1'' = 0$ and that $t^2y_1'' + 2ty_1' - 2y_1 = t^2(0) + 2t(1) - 2(t) = 2t - 2t = 0$. Hence $y_1(t) = t$ solves the ODE.

(ii) As per the hint, we start with $y_2(t) = v(t)y_1(t) = v(t)t$. Then $y_2' = v't+v$ and $y_2'' = v''t+2v'$. Substituting into the ODE, we calculate that

$$\begin{aligned} 0 &= t^2y_2'' + 2ty_2' - 2y_2 \\ &= t^2(v''t + 2v') + 2t(v't + v) - 2vt \\ &= t^3v'' + v'(2t^2 + 2t^2) + v(2t - 2t) \\ &= t^3v'' + 4t^2v' \\ &= t^2(tv'' + 4v'). \end{aligned}$$

Letting $u = v'$, we obtain the first order ODE

$$t\frac{du}{dt} + 4u = 0.$$

We calculate that

$$\begin{aligned} t\frac{du}{dt} &= -4u \\ \frac{du}{u} &= -4\frac{dt}{t} \\ \int \frac{du}{u} &= -4\int \frac{dt}{t} \\ \ln|u| &= -4\ln|t| + C \\ u &= \pm e^Ct^{-4} = ct^{-4} \end{aligned}$$

and

$$v = \int u\;dt = \int ct^{-4}\;dt = -\frac{1}{3}ct^{-3} + k.$$

Thus $y_2(t) = v(t)t = -\frac{1}{3}ct^{-2} + kt$. Choosing $c = -3$ and $k = 0$, we obtain the solution

$$y_2(t) = t^{-2}.$$

(iii) Since $y_2' = -2t^{-3}$ and $y_2'' = 6t^{-4}$, we have that

$$\begin{aligned} t^2y_2'' + 2ty_2' - 2y_2 &= t^2(6t^{-4}) + 2t(-2t^{-3}) - 2t^{-2} \\ &= 6t^{-2} - 4t^{-2} - 2t^{-2} \\ &= 0 \end{aligned}$$

as required.

(iv) We have that

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & t^{-2} \\ 1 & -2t^{-3} \end{vmatrix} = -2t^{-2} - t^{-2} = -3t^{-2} \neq 0.$$

Therefore y_1 and y_2 are linearly independent.

(b) $y_2(t) = t^3$

(c) $y_2(t) = t^{-1}\ln t$

(d) $y_2(t) = te^t$

(e) $y_2(x) = \cos x^2$

(f) $y_2(x) = x$