Exercise 30 (Systems of Linear Equations). Find the general solutions to the following systems of ODEs:

(a) $\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{x}$

(b) $\mathbf{x}' = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} \mathbf{x}$

(c) $\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \mathbf{x}$.

(d) $\begin{cases} x' = 4x - y \\ y' = x + 2y \end{cases}$

(e) $\begin{cases} x' = 3x - y \\ y' = 4x - y \end{cases}$

(f) $\begin{cases} x' = 5x + 4y \\ y' = -x + y \end{cases}$

(g) $\begin{cases} x' = 3x + 2y \\ y' = -5x + y \end{cases}$

(h) $\begin{cases} x' = x - 4y \\ y' = x + y \end{cases}$

(i) $\begin{cases} x' = x - 3y \\ y' = 3x + y \end{cases}$

(j) $\begin{cases} x' = 4x - 2y \\ y' = 5x + 2y \end{cases}$

(k) $\begin{cases} x' = x + y - z \\ y' = 2x + 3y - 4z \\ z' = 4x + y - 4z \end{cases}$

(l) $\begin{cases} x' = x - y - z \\ y' = x + 3y + z \\ z' = -3x + y - z \end{cases}$

(m) $\begin{cases} x' = 3x + y + z \\ y' = 3y + z \\ z' = 6z \end{cases}$

(n) $\begin{cases} x' = 2x + y - z \\ y' = -4x - 3y - z \\ z' = 4x + 4y + 2z \end{cases}$

Solution 30.

(a) Note that

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1).$$

Thus, the eigenvalues of $\mathbf{A}$ are $\{3, -1\}$. Since the eigenvalues of $\mathbf{A}$ are real and distinct, the eigenvectors of $\mathbf{A}$ are linearly independent and can be calculated as follows.

$$\begin{aligned} \mathbf{0} &= (\mathbf{A} - 3\mathbf{I})\,\mathbf{q}_1 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \mathbf{0} &= (\mathbf{A} + \mathbf{I})\,\mathbf{q}_2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \end{aligned}$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1 \mathbf{q}_1 e^{3t} + c_2 \mathbf{q}_2 e^{-t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}, \\ &= \begin{bmatrix} c_1 e^{3t} + c_2 e^{-t} \\ c_1 e^{3t} - c_2 e^{-t} \end{bmatrix}. \end{aligned}$$

(b) Note that

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -3-\lambda & 2 \\ -3 & 4-\lambda \end{vmatrix} = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2).$$

Thus, the eigenvalues of $\mathbf{A}$ are $\{3, -2\}$. Since the eigenvalues of $\mathbf{A}$ are real and distinct, the eigenvectors of $\mathbf{A}$ are linearly independent and can be calculated as follows.

$$\begin{aligned} \mathbf{0} &= (\mathbf{A} - 3\mathbf{I})\,\mathbf{q}_1 = \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \\ \mathbf{0} &= (\mathbf{A} + 2\mathbf{I})\,\mathbf{q}_2 = \begin{bmatrix} -1 & 2 \\ -3 & 6 \end{bmatrix} \mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \end{aligned}$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1 \mathbf{q}_1 e^{3t} + c_2 \mathbf{q}_2 e^{-2t} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-2t}, \\ &= \begin{bmatrix} c_1 e^{3t} + 2c_2 e^{-2t} \\ 3c_1 e^{3t} + c_2 e^{-2t} \end{bmatrix}. \end{aligned}$$

(c) Note that

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 3-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix} = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2).$$

Thus, the eigenvalues of $\mathbf{A}$ are $\{2, -2\}$. Since the eigenvalues of $\mathbf{A}$ are real and distinct, the eigenvectors of $\mathbf{A}$ are linearly independent and can be calculated as follows.

$$\begin{aligned} \mathbf{0} &= (\mathbf{A} - 2\mathbf{I})\,\mathbf{q}_1 = \begin{bmatrix} 1 & -1 \\ 5 & -5 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \mathbf{0} &= (\mathbf{A} + 2\mathbf{I})\,\mathbf{q}_2 = \begin{bmatrix} 5 & -1 \\ 5 & -1 \end{bmatrix} \mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}. \end{aligned}$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1 \mathbf{q}_1 e^{2t} + c_2 \mathbf{q}_2 e^{-2t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{-2t}, \\ &= \begin{bmatrix} c_1 e^{2t} + c_2 e^{-2t} \\ c_1 e^{2t} + 5c_2 e^{-2t} \end{bmatrix}. \end{aligned}$$

(d) The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus, the eigenvalues of $\mathbf{A}$ are

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 4-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2.$$

In this case, we first calculate the generalized eigenvector $\mathbf{q}_1$ of order one as follows.

$$\mathbf{0} = (\mathbf{A} - 3\mathbf{I})\,\mathbf{q}_1 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then, we calculate the generalized eigenvector $\mathbf{r}_1$ of order two as follows.

$$(\mathbf{A} - 3\mathbf{I})\,\mathbf{r}_1 = \mathbf{q}_1 \Longrightarrow \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \mathbf{r}_1 = \mathbf{q}_1 \Longrightarrow \mathbf{r}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{q}_1 e^{3t} + c_2(t\mathbf{q}_1 + \mathbf{r}_1)e^{3t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2\left(t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)e^{3t}, \\ &= \begin{bmatrix} c_1 e^{3t} + c_2(t+1)e^{3t} \\ c_1 e^{3t} + c_2 t e^{3t} \end{bmatrix}. \end{aligned}$$

(e) The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus, the eigenvalues of $\mathbf{A}$ are

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & -1 \\ 4 & -1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2.$$

Similar to the previous case, we first calculate the generalized eigenvector $\mathbf{q}_1$ of order one as follows.

$$\mathbf{0} = (\mathbf{A} - \mathbf{I})\,\mathbf{q}_1 = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Then, we calculate the generalized eigenvector $\mathbf{r}_1$ of order two as follows.

$$(\mathbf{A} - \mathbf{I})\,\mathbf{r}_1 = \mathbf{q}_1 \Longrightarrow \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \mathbf{r}_1 = \mathbf{q}_1 \Longrightarrow \mathbf{r}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{q}_1 e^{t} + c_2(t\mathbf{q}_1 + \mathbf{r}_1)e^{t} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{t} + c_2\left(t\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)e^{t}, \\ &= \begin{bmatrix} c_1 e^{t} + c_2 t e^{t} \\ 2c_1 e^{t} + c_2(2t-1)e^{t} \end{bmatrix}. \end{aligned}$$

(f) The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus, the eigenvalues of $\mathbf{A}$ are

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 5-\lambda & 4 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2.$$

Similar to the previous case, we first calculate the generalized eigenvector $\mathbf{q}_1$ of order one as follows.

$$\mathbf{0} = (\mathbf{A} - 3\mathbf{I})\,\mathbf{q}_1 = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Then, we calculate the generalized eigenvector $\mathbf{r}_1$ of order two as follows.

$$(\mathbf{A} - 3\mathbf{I})\,\mathbf{r}_1 = \mathbf{q}_1 \Longrightarrow \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \mathbf{r}_1 = \mathbf{q}_1 \Longrightarrow \mathbf{r}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{q}_1 e^{3t} + c_2(t\mathbf{q}_1 + \mathbf{r}_1)e^{3t} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{3t} + c_2\left(t\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)e^{3t}, \\ &= \begin{bmatrix} 2c_1 e^{3t} + c_2(2t+1)e^{3t} \\ -c_1 e^{3t} - c_2 t e^{3t} \end{bmatrix}. \end{aligned}$$

(g) The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 2 \\ -5 & 1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 13.$$

Note that the characteristic polynomial is of the form $\lambda^2 - 2\sigma\lambda + \sigma^2 + w^2$ where $\sigma = 2$ and $w = 3$. Thus, the eigenvalues of $\mathbf{A}$ are $\{2 \pm 3j\}$. The eigenvectors are $\mathbf{q}_1 \pm j\mathbf{q}_2$ which satisfy the following equation.

$$\begin{aligned} \mathbf{A}\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \Longrightarrow \\ (\mathbf{A} - 2\mathbf{I})\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}. \end{aligned}$$

Since $\mathbf{A}^2 - 2\sigma\mathbf{A} + (\sigma^2 + w^2)\,\mathbf{I} = \mathbf{0}$, $\mathbf{q}_1$ can be chosen as an arbitrary nonzero vector. Let us choose $\mathbf{q}_1 := \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and calculate $\mathbf{q}_2$ as follows

$$(\mathbf{A} - 2\mathbf{I})\,\mathbf{q}_1 = -3\mathbf{q}_2 \Longrightarrow \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} \mathbf{q}_1 = -3\mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= (c_1\mathbf{q}_1 + c_2\mathbf{q}_2)e^{2t}\cos 3t + (c_2\mathbf{q}_1 - c_1\mathbf{q}_2)e^{2t}\sin 3t, \\ &= \left(c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)e^{2t}\cos 3t + \left(c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} - c_1\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)e^{2t}\sin 3t \\ &= e^{2t}\begin{bmatrix} (c_1 - c_2)\cos 3t + (c_1 + c_2)\sin 3t \\ (c_1 + 2c_2)\cos 3t - (2c_1 - c_2)\sin 3t \end{bmatrix}. \end{aligned}$$

(h) The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 1-\lambda & -4 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 5.$$

As in the previous case, the characteristic polynomial is of the form $\lambda^2 - 2\sigma\lambda + \sigma^2 + w^2$ where $\sigma = 1$ and $w = 2$. Thus, the eigenvalues of $\mathbf{A}$ are $\{1 \pm 2j\}$. The eigenvectors are $\mathbf{q_1} \pm j\mathbf{q_2}$ which satisfy the following equation.

$$\begin{aligned} \mathbf{A} \begin{bmatrix} \mathbf{q_1} & \mathbf{q_2} \end{bmatrix} &= \begin{bmatrix} \mathbf{q_1} & \mathbf{q_2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \Longrightarrow \\ (\mathbf{A} - \mathbf{I}) \begin{bmatrix} \mathbf{q_1} & \mathbf{q_2} \end{bmatrix} &= \begin{bmatrix} \mathbf{q_1} & \mathbf{q_2} \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}. \end{aligned}$$

Since $\mathbf{A}^2 - 2\sigma\mathbf{A} + (\sigma^2 + w^2)\,\mathbf{I} = \mathbf{0}$, $\mathbf{q_1}$ can be chosen as an arbitrary nonzero vector. Let us choose $\mathbf{q_1} := \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ and calculate $\mathbf{q_2}$ as follows

$$(\mathbf{A} - \mathbf{I})\,\mathbf{q}_1 = -2\mathbf{q}_2 \Longrightarrow \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix} \mathbf{q}_1 = -2\mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= (c_1\mathbf{q}_1 + c_2\mathbf{q}_2)\, e^t \cos 2t + (c_2\mathbf{q}_1 - c_1\mathbf{q}_2)\, e^t \sin 2t, \\ &= \left(c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) e^t \cos 2t + \left(c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} - c_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) e^t \sin 2t \\ &= e^t \begin{bmatrix} 2\,(c_2 \cos 2t - c_1 \sin 2t) \\ c_1 \cos 2t + c_2 \sin 2t \end{bmatrix}. \end{aligned}$$

(i) The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 1-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 10.$$

As in the previous case, the characteristic polynomial is of the form $\lambda^2 - 2\sigma\lambda + \sigma^2 + w^2$ where $\sigma = 1$ and $w = 3$. Thus, the eigenvalues of $\mathbf{A}$ are $\{1 \pm 3j\}$. The eigenvectors are $\mathbf{q_1} \pm j\mathbf{q_2}$ which satisfy the following equation.

$$\begin{aligned} \mathbf{A} \begin{bmatrix} \mathbf{q_1} & \mathbf{q_2} \end{bmatrix} &= \begin{bmatrix} \mathbf{q_1} & \mathbf{q_2} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \Longrightarrow \\ (\mathbf{A} - \mathbf{I}) \begin{bmatrix} \mathbf{q_1} & \mathbf{q_2} \end{bmatrix} &= \begin{bmatrix} \mathbf{q_1} & \mathbf{q_2} \end{bmatrix} \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}. \end{aligned}$$

Since $\mathbf{A}^2 - 2\sigma\mathbf{A} + (\sigma^2 + w^2)\,\mathbf{I} = \mathbf{0}$, $\mathbf{q_1}$ can be chosen as an arbitrary nonzero vector. Let us choose $\mathbf{q_1} := \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and calculate $\mathbf{q_2}$ as follows

$$(\mathbf{A} - \mathbf{I})\,\mathbf{q}_1 = -3\mathbf{q}_2 \Longrightarrow \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \mathbf{q}_1 = -3\mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= (c_1\mathbf{q}_1 + c_2\mathbf{q}_2)\, e^t \cos 3t + (c_2\mathbf{q}_1 - c_1\mathbf{q}_2)\, e^t \sin 3t, \\ &= \left(c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^t \cos 3t + \left(c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^t \sin 3t \\ &= e^t \begin{bmatrix} (c_1 \cos 3t + c_2 \sin 3t) \\ (c_2 \cos 3t - c_1 \sin 3t) \end{bmatrix}. \end{aligned}$$

(j) The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 4 & -2 \\ 5 & 2 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 4-\lambda & -2 \\ 5 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 18.$$

As in the previous case, the characteristic polynomial is of the form $\lambda^2 - 2\sigma\lambda + \sigma^2 + w^2$ where $\sigma = 3$ and $w = 3$. Thus, the eigenvalues of $\mathbf{A}$ are $\{3 \pm 3j\}$. The eigenvectors are $\mathbf{q_1} \pm j\mathbf{q_2}$ which satisfy the following equation.

$$\begin{aligned} \mathbf{A} \begin{bmatrix} \mathbf{q_1} & \mathbf{q_2} \end{bmatrix} &= \begin{bmatrix} \mathbf{q_1} & \mathbf{q_2} \end{bmatrix} \begin{bmatrix} 3 & 3 \\ -3 & 3 \end{bmatrix} \Longrightarrow \\ (\mathbf{A} - 3\mathbf{I}) \begin{bmatrix} \mathbf{q_1} & \mathbf{q_2} \end{bmatrix} &= \begin{bmatrix} \mathbf{q_1} & \mathbf{q_2} \end{bmatrix} \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}. \end{aligned}$$

Since $\mathbf{A}^2 - 2\sigma\mathbf{A} + (\sigma^2 + w^2)\,\mathbf{I} = \mathbf{0}$, $\mathbf{q_1}$ can be chosen as an arbitrary nonzero vector. Let us choose $\mathbf{q_1} := \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and calculate $\mathbf{q_2}$ as follows

$$(\mathbf{A} - 3\mathbf{I})\,\mathbf{q}_1 = -3\mathbf{q}_2 \Longrightarrow \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \mathbf{q}_1 = -3\mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= (c_1\mathbf{q}_1 + c_2\mathbf{q}_2)\, e^{3t} \cos 3t + (c_2\mathbf{q}_1 - c_1\mathbf{q}_2)\, e^{3t} \sin 3t, \\ &= \left(c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{3t} \cos 3t + \left(c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{3t} \sin 3t \\ &= e^{3t} \begin{bmatrix} c_1 \cos 3t + c_2 \sin 3t \\ c_2 \cos 3t - c_1 \sin 3t \end{bmatrix}. \end{aligned}$$

(k) The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 2 & 3-\lambda & -4 \\ 4 & 1 & -4-\lambda \end{vmatrix} = (\lambda-1)(\lambda-2)(\lambda+3).$$

Thus, the eigenvalues of $\mathbf{A}$ are $\{1, 2, -3\}$. Since the eigenvalues are real and distinct, the eigenvectors of $\mathbf{A}$ are linearly independent and can be calculated as follows.

$$\begin{aligned} \mathbf{0} &= (\mathbf{A}-\mathbf{I})\,\mathbf{q}_1 = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 2 & -4 \\ 4 & 1 & -5 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \\ \mathbf{0} &= (\mathbf{A}-2\mathbf{I})\,\mathbf{q}_2 = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 1 & -4 \\ 4 & 1 & -6 \end{bmatrix} \mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \\ \mathbf{0} &= (\mathbf{A}+3\mathbf{I})\,\mathbf{q}_3 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 6 & -4 \\ 4 & 1 & -1 \end{bmatrix} \mathbf{q}_3 \Longrightarrow \mathbf{q}_3 = \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix}. \end{aligned}$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{q}_1e^{t} + c_2\mathbf{q}_2e^{2t} + c_3\mathbf{q}_3e^{-3t} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix} e^{-3t}, \\ &= \begin{bmatrix} c_1e^{t} + c_2e^{2t} + c_3e^{-3t} \\ c_1e^{t} + 2c_2e^{2t} + 7c_3e^{-3t} \\ c_1e^{t} + c_2e^{2t} + 11c_3e^{-3t} \end{bmatrix}. \end{aligned}$$

(l) The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -1 & -1 \\ 1 & 3-\lambda & 1 \\ -3 & 1 & -1-\lambda \end{vmatrix} = (\lambda-2)(\lambda-3)(\lambda+2).$$

Thus, the eigenvalues of $\mathbf{A}$ are $\{2, 3, -2\}$. Since the eigenvalues are real and distinct, the eigenvectors of $\mathbf{A}$ are linearly independent and can be calculated as follows.

$$\begin{aligned} \mathbf{0} &= (\mathbf{A}-2\mathbf{I})\,\mathbf{q}_1 = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -3 & 1 & -3 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \\ \mathbf{0} &= (\mathbf{A}-3\mathbf{I})\,\mathbf{q}_2 = \begin{bmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ -3 & 1 & -4 \end{bmatrix} \mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \\ \mathbf{0} &= (\mathbf{A}+2\mathbf{I})\,\mathbf{q}_3 = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 5 & 1 \\ -3 & 1 & 1 \end{bmatrix} \mathbf{q}_3 \Longrightarrow \mathbf{q}_3 = \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix}. \end{aligned}$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{q}_1e^{2t} + c_2\mathbf{q}_2e^{3t} + c_3\mathbf{q}_3e^{-2t} = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix} e^{-2t}, \\ &= \begin{bmatrix} c_1e^{2t} + c_2e^{3t} - c_3e^{-2t} \\ -c_2e^{2t} + c_3e^{-2t} \\ -c_1e^{t} - c_2e^{2t} - 4c_3e^{-2t} \end{bmatrix}. \end{aligned}$$

(m) The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 6 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 6-\lambda \end{vmatrix} = (\lambda-6)(\lambda-3)^2.$$

Thus, the eigenvalues of $\mathbf{A}$ are $\{3, 3, 6\}$. The eigenvectors belonging to $\lambda = 3$ are generalized eigenvectors and can be calculated as in Question 6. Therefore, it follows that

$$\begin{aligned} \mathbf{0} &= (\mathbf{A}-3\mathbf{I})\,\mathbf{q}_1 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \\ (\mathbf{A}-3\mathbf{I})\,\mathbf{r}_1 &= \mathbf{q}_1 \Longrightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{r}_1 = \mathbf{q}_1 \Longrightarrow \mathbf{r}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \\ \mathbf{0} &= (\mathbf{A}-6\mathbf{I})\,\mathbf{q}_2 = \begin{bmatrix} -3 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 4 \\ 3 \\ 9 \end{bmatrix}. \end{aligned}$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{q}_1e^{3t} + c_2(t\mathbf{q}_1 + \mathbf{r}_1)e^{3t} + c_3\mathbf{q}_2e^{6t} \\ &= c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{3t} + c_2 \left(t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) e^{3t} + c_3 \begin{bmatrix} 4 \\ 3 \\ 9 \end{bmatrix} e^{6t}, \\ &= \begin{bmatrix} c_1e^{3t} + c_2te^{3t} + 4c_3e^{6t} \\ c_2e^{3t} + 3c_3e^{6t} \\ 9c_3e^{6t} \end{bmatrix}. \end{aligned}$$

(n) The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 2-\lambda & 1 & -1 \\ -4 & -3-\lambda & -1 \\ 4 & 4 & 2-\lambda \end{vmatrix} = (\lambda - 1)\left(\lambda^2 + 4\right).$$

Thus, the eigenvalues of $\mathbf{A}$ are $\{1, \pm 2j\}$. The eigenvectors belonging to $\lambda = \pm 2j$ are $\{\mathbf{q}_1 \pm j\mathbf{q}_2\}$.Note that

$$\mathbf{A}^2 + 4\mathbf{I} = \begin{bmatrix} -4 & -5 & -5 \\ 0 & 1 & 5 \\ 0 & 0 & -4 \end{bmatrix} + 4\mathbf{I} = \begin{bmatrix} 0 & -5 & -5 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can choose $\mathbf{q}_1$ so that $\left(\mathbf{A}^2 + 4\mathbf{I}\right)\mathbf{q}_1 = \mathbf{0}$. Hence, we get

$$\mathbf{0} = \left(\mathbf{A}^2 + 4\mathbf{I}\right)\mathbf{q}_1 = \begin{bmatrix} 0 & -5 & -5 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then, similar to the problems in $\mathbb{R}^2$, we can choose $\mathbf{q}_2$as follows.

$$(\mathbf{A} - \sigma\mathbf{I})\mathbf{q}_1 = -w\mathbf{q}_2 \Longrightarrow \begin{bmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{bmatrix} \mathbf{q}_1 = -2\mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}.$$

Finally, the eigenvector for $\lambda = 1$ can be calculated as follows.

$$\mathbf{0} = (\mathbf{A} - \mathbf{I})\mathbf{q}_3 = \begin{bmatrix} 1 & 1 & -1 \\ -4 & -4 & -1 \\ 4 & 4 & 1 \end{bmatrix} \mathbf{q}_3 \Longrightarrow \mathbf{q}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= (c_1\mathbf{q}_1 + c_2\mathbf{q}_2)\cos 2t + (c_2\mathbf{q}_1 - c_1\mathbf{q}_2)\sin 2t + c_3\mathbf{q}_3 e^t \\ &= \left(c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}\right)\cos 2t + \left(c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - c_1 \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}\right)\sin 2t + c_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^t, \\ &= \begin{bmatrix} (c_1 - c_2)\cos 2t + (c_1 + c_2)\sin 2t + c_3 e^t \\ 2c_2\cos 2t - 2c_1 \sin 2t - c_3 e^t \\ 2c_1 \sin 2t - 2c_2 \cos 2t \end{bmatrix}. \end{aligned}$$

Exercise 31 (Initial Value Problems). Solve the following IVPs:

(a) $\begin{cases} \mathbf{x}' = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{cases}$ (b) $\begin{cases} \mathbf{x}' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 8 \\ 0 \end{bmatrix} \end{cases}$ (c) $\begin{cases} x' = 3x + z \\ y' = 9x - y + 2z \\ z' = -9x + 4y - z \\ x(0) = 0 \\ y(0) = 0 \\ z(0) = 17 \end{cases}$

Solution 31.

(a) Note that

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 3-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 5\lambda - 6 = (\lambda - 6)(\lambda + 1).$$

Thus, the eigenvalues of $\mathbf{A}$ are $\{6, -1\}$. Since the eigenvalues of $\mathbf{A}$ are real and distinct, the eigenvectors of $\mathbf{A}$ are linearly independent and can be calculated as follows.

$$\begin{aligned} \mathbf{0} &= (\mathbf{A} - 6\mathbf{I})\,\mathbf{q}_1 = \begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \\ \mathbf{0} &= (\mathbf{A} + \mathbf{I})\,\mathbf{q}_2 = \begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix} \mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \end{aligned}$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1 \mathbf{q}_1 e^{6t} + c_2 \mathbf{q}_2 e^{-t} = c_1 \begin{bmatrix} 4 \\ 3 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}, \\ &= \begin{bmatrix} c_1 e^{6t} + c_2 e^{-t} \\ c_1 e^{6t} - c_2 e^{-t} \end{bmatrix}. \end{aligned}$$

Note that at $t = 0$, we have

$$\begin{aligned} \mathbf{x}(0) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Longrightarrow \\ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \frac{1}{7} \begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 2 \\ -1 \end{bmatrix}. \end{aligned}$$

Thus, the solution of the initial value problem is

$$\mathbf{x}(t) = \frac{1}{7} \begin{bmatrix} 2e^{6t} - e^{-t} \\ 2e^{3t} + e^{-t} \end{bmatrix}.$$

(b) Note that

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 4-\lambda & -3 \\ 6 & -7-\lambda \end{vmatrix} = \lambda^2 + 3\lambda - 10 = (\lambda - 2)(\lambda + 5).$$

Thus, the eigenvalues of $\mathbf{A}$ are $\{2, -5\}$. Since the eigenvalues of $\mathbf{A}$ are real and distinct, the eigenvectors of $\mathbf{A}$ are linearly independent and can be calculated as follows.

$$\begin{aligned} \mathbf{0} &= (\mathbf{A} - 2\mathbf{I})\,\mathbf{q}_1 = \begin{bmatrix} 2 & -3 \\ 6 & -9 \end{bmatrix} \mathbf{q}_1 \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \\ \mathbf{0} &= (\mathbf{A} + 5\mathbf{I})\,\mathbf{q}_2 = \begin{bmatrix} 9 & -3 \\ 6 & -2 \end{bmatrix} \mathbf{q}_2 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}. \end{aligned}$$

Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1 \mathbf{q}_1 e^{2t} + c_2 \mathbf{q}_2 e^{-5t} = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-5t}, \\ &= \begin{bmatrix} 3c_1 e^{2t} + c_2 e^{-5t} \\ 2c_1 e^{2t} + 3c_2 e^{-5t} \end{bmatrix}. \end{aligned}$$

Note that at $t = 0$, we have

$$\begin{aligned} \mathbf{x}(0) &= \begin{bmatrix} 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Longrightarrow \\ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \frac{1}{7} \begin{bmatrix} 3 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 24 \\ -16 \end{bmatrix}. \end{aligned}$$

Thus, the solution of the initial value problem is

$$\mathbf{x}(t) = \frac{8}{7} \begin{bmatrix} 9e^{2t} - 2e^{-5t} \\ 6e^{2t} - 6e^{-5t} \end{bmatrix}.$$

(c) The equations above can be written as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 3 & 0 & 1 \\ 9 & -1 & 2 \\ -9 & 4 & -1 \end{bmatrix} \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 9 & -1-\lambda & 2 \\ -9 & 4 & -1-\lambda \end{vmatrix} = \left(\lambda^2 + 2\lambda + 2\right)(\lambda - 3).$$

Thus, the eigenvalues of $\mathbf{A}$ are $\{3, -1 \pm j\}$. The eigenvectors belonging to $\lambda = -1 \pm j$ are $\{\mathbf{q}_1 \pm j\mathbf{q}_2\}$.Note that

$$\mathbf{A}^2 + 2\mathbf{A} + 2\mathbf{I} = \begin{bmatrix} 8 & 4 & 4 \\ 18 & 9 & 9 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can choose $\mathbf{q}_1$ so that $\left(\mathbf{A}^2+2\mathbf{A}+2\mathbf{I}\right)\mathbf{q}_1=\mathbf{0}$. Hence, we get

$$\mathbf{0}=\left(\mathbf{A}^2+2\mathbf{A}+2\mathbf{I}\right)\mathbf{q}_1=\begin{bmatrix}8&4&4\\18&9&9\\0&0&0\end{bmatrix}\mathbf{q}_1\Longrightarrow\mathbf{q}_1=\begin{bmatrix}1\\-1\\-1\end{bmatrix}.$$

Then, similar to the problems in $\mathbb{R}^2$, we can choose $\mathbf{q}_2$as follows.

$$(\mathbf{A}+\mathbf{I})\,\mathbf{q}_1=-\mathbf{q}_2\Longrightarrow\begin{bmatrix}4&0&1\\9&0&2\\-9&4&0\end{bmatrix}\mathbf{q}_1=-\mathbf{q}_2\Longrightarrow\mathbf{q}_2=\begin{bmatrix}-3\\-7\\13\end{bmatrix}.$$

Finally, the eigenvector for $\lambda=3$ can be calculated as follows.

$$\mathbf{0}=(\mathbf{A}-3\mathbf{I})\,\mathbf{q}_3=\begin{bmatrix}0&0&1\\9&-4&2\\-9&4&-4\end{bmatrix}\mathbf{q}_3\Longrightarrow\mathbf{q}_3=\begin{bmatrix}4\\9\\0\end{bmatrix}.$$

Consequently, the general solution is

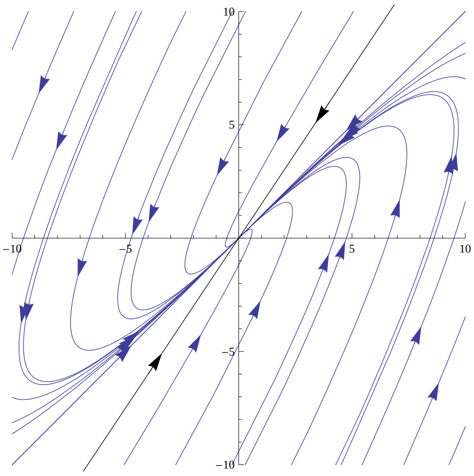
$$\begin{aligned}\mathbf{x}(t)&=(c_1\mathbf{q}_1+c_2\mathbf{q}_2)\,e^{-t}\cos t+(c_2\mathbf{q}_1-c_1\mathbf{q}_2)\,e^{-t}\sin t+c_3\mathbf{q}_3e^{3t}\\&=\left(c_1\begin{bmatrix}1\\-1\\-1\end{bmatrix}+c_2\begin{bmatrix}-3\\-7\\13\end{bmatrix}\right)e^{-t}\cos t+\left(c_2\begin{bmatrix}1\\-1\\-1\end{bmatrix}-c_1\begin{bmatrix}-3\\-7\\13\end{bmatrix}\right)e^{-t}\sin t+c_3\begin{bmatrix}4\\9\\0\end{bmatrix}e^{3t}\\&=e^{-t}\begin{bmatrix}(c_1-3c_2)\cos t+(3c_1+c_2)\sin t+4c_3e^{4t}\\-(c_1+7c_2)\cos t+(7c_1-c_2)\sin t+9c_3e^{4t}\\(13c_2-c_1)\cos t-(13c_1+c_2)\sin t\end{bmatrix}.\end{aligned}$$

Note that at $t=0$, we get the solution for the initial value problem.

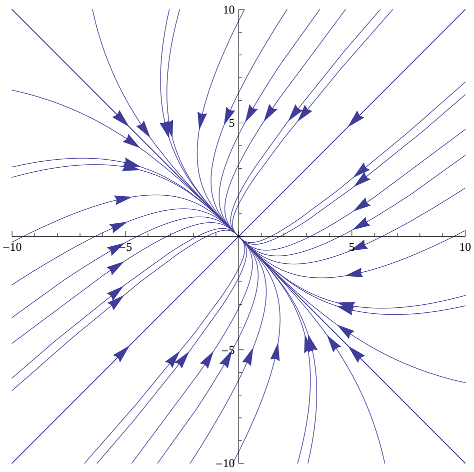
$$\begin{aligned}\begin{bmatrix}0\\0\\17\end{bmatrix}&=\begin{bmatrix}1&-3&4\\-1&-7&9\\-1&13&0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}\Longrightarrow\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\frac{1}{170}\begin{bmatrix}117&-52&-1\\9&-4&13\\20&10&10\end{bmatrix}\begin{bmatrix}0\\0\\17\end{bmatrix},\\\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}&=\frac{1}{10}\begin{bmatrix}-1\\13\\10\end{bmatrix}\Longrightarrow\mathbf{x}(t)=e^{-t}\begin{bmatrix}-4\cos t+\sin t+4e^{4t}\\-9\cos t-2\sin t+9e^{4t}\\17\cos t\end{bmatrix}.\end{aligned}$$

Changing back to the notation in the question, we have that

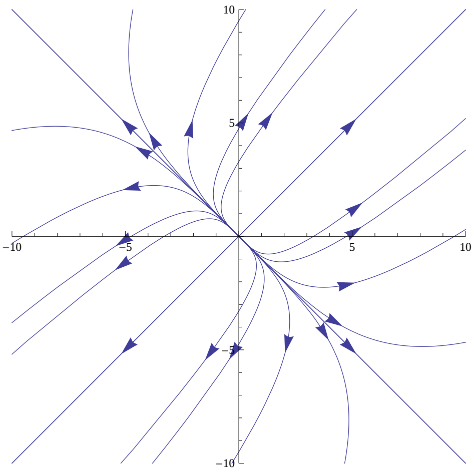
$$\begin{aligned}x(t)&=-4e^{-t}\cos t+e^{-t}\sin t+4e^{3t}\\y(t)&=-9e^{-t}\cos t-2e^{-t}\sin t+9e^{3t}\\z(t)&=17e^{-t}\cos t.\end{aligned}$$



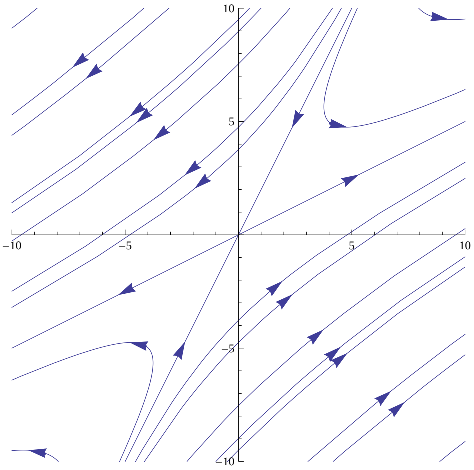
(i) Stable node



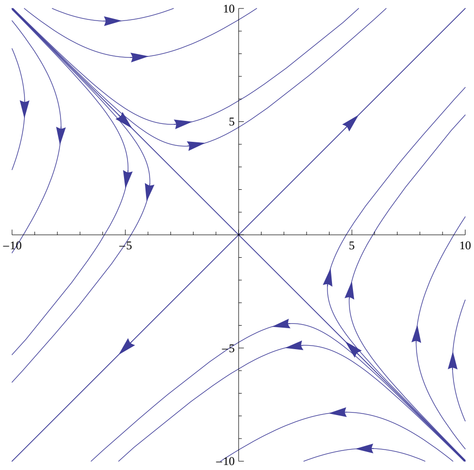
(ii) Stable node



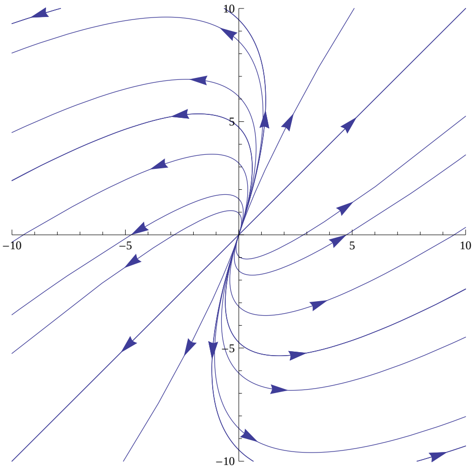
(iii) Unstable node



(iv) Saddle



(v) Saddle



(vi) Unstable node

Exercise 32 (Phase Plots)**.** Match the system of ODEs $\mathbf{x}' = A\mathbf{x}$ with the correct phase plot shown above, for each matrix A below. You are given eigenvalues and eigenvectors for each matrix. The first one is done for you.

(ω) $A = \begin{bmatrix} 3 & 5 \\ 5 & 3 \end{bmatrix}$; $r_1 = 8,\ r_2 = -2$; $\xi^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \xi^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

> Since one eigenvalue is positive and one is negative, $\mathbf{x}' = A\mathbf{x}$ must have a saddle point. So the phase plot must be either (iv) or (v).
>
> The phase plot must also have straight lines in the directions of the eigenvectors. So it must be (v).

(a) $A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$; $r_1 = -2,\ r_2 = -1$; $\xi^{(1)} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \xi^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$; $r_1 = -3,\ r_2 = -1$; $\xi^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \xi^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$; $r_1 = 4,\ r_2 = 2$; $\xi^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \xi^{(2)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

(d) $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$; $r_1 = 8,\ r_2 = 2$; $\xi^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \xi^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(e) $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$; $r_1 = 2,\ r_2 = -1$; $\xi^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \xi^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Solution 32.

(a) i (b) ii (c) vi (d) iii (e) iv