

Exercise 33 (Fundamental Matrices).

- (a) Suppose that $\Psi(t)$ is a fundamental matrix for $\mathbf{x}' = A\mathbf{x}$, where $A \in \mathbb{R}^{n \times n}$. Show that $\Psi' = A\Psi$.
- (b) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Find $e^{At} = \exp(At)$.

For each of the following;

- (i) Find a fundamental matrix $\Psi(t)$ for the system; and
- (ii) Find the special fundamental matrix $\Phi(t)$ which satisfies $\Phi(0) = I$.

The first one is done for you.

$$\begin{array}{lll} (\omega) \mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} & (e) \mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x} & (h) \mathbf{x}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x} \\ (c) \mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x} & (f) \mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{x} & \\ (d) \mathbf{x}' = \begin{bmatrix} -\frac{3}{4} & \frac{1}{8} \\ \frac{1}{8} & -\frac{3}{4} \end{bmatrix} \mathbf{x} & (g) \mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x} & (i) \mathbf{x}' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{bmatrix} \mathbf{x} \end{array}$$

(ω) (i) The general solution to this system is $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$. Therefore $\Psi(t) = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix}$ is a fundamental matrix for this system.

(ii) We must solve $\begin{cases} \mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{cases}$ and $\begin{cases} \mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} \\ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$. Using the general solution from above, we calculate that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \implies c_1 = c_2 = \frac{1}{2} \implies \mathbf{x}^{(1)}(t) = \begin{bmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \\ e^{3t} - e^{-t} \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \implies c_1 = \frac{1}{4}, c_2 = -\frac{1}{4} \implies \mathbf{x}^{(2)}(t) = \begin{bmatrix} \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{bmatrix}$$

Therefore

$$\Phi(t) = \begin{bmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{bmatrix}.$$

Exercise 34 (Non-Homogeneous Systems of Equations). Solve the following systems of ODEs:

- (a) $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ t \end{bmatrix}$
- (b) $\mathbf{x}' = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ \sqrt{3}e^{-t} \end{bmatrix}$
- (c) $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^{-2t} \\ -2e^t \end{bmatrix}$
- (d) $\mathbf{x}' = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t^{-1} \\ 2t^{-1} + 4 \end{bmatrix}, \quad t > 0$
- (e) $\mathbf{x}' = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t^{-3} \\ -t^{-2} \end{bmatrix}, \quad t > 0$