MATH 216 MATHEMATICS IV
Homework 2 : Solutions

A. Find the general solutions of the following homogeneous differential equations:

1. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0.$

Solution : The characteristic equation is

$$\lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0.$$

The roots of the characteristic equation are $\{-1, 3\}$. Consequently, the solution of the homogeneous equation is

$$y_H(x) = c_1 e^{-x} + c_2 e^{3x}.$$

2. $4\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 5y = 0.$

Solution : The standard form of this equation is

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + \frac{5}{4}y = 0.$$

The characteristic equation is $\lambda^2 - 3\lambda + \frac{5}{4} = \left(\lambda - \frac{5}{2}\right)\left(\lambda - \frac{1}{2}\right) = 0$. The roots of the characteristic equation are $\left\{\frac{5}{2}, \frac{1}{2}\right\}$. Consequently, the solution of the homogeneous equation is

$$y_H(x) = c_1 e^{\frac{x}{2}} + c_2 e^{\frac{5x}{2}}.$$

3. $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0.$

Solution : The characteristic equation is $\lambda^3 - 4\lambda^2 + \lambda + 6 = (\lambda + 1)(\lambda - 2)(\lambda - 3)$. The roots of the characteristic equation are $\{-1, 2, 3\}$. Consequently, the solution of the homogeneous equation is

$$y_H(x) = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{3x}.$$

4. $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0.$

Solution : The characteristic equation is $\lambda^2 - 8\lambda + 16 = (\lambda - 4)^2$. The roots of the characteristic equation are $\{4, 4\}$. Consequently, the solution of the homogeneous equation is

$$y_H(x) = c_1 e^{4x} + c_2 x e^{4x}.$$

5. $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 18y = 0.$

Solution : The characteristic equation is $\lambda^3 - 4\lambda^2 - 3\lambda + 18 = (\lambda + 2)(\lambda - 3)^2$. The roots of the characteristic equation are $\{-2, 3, 3\}$. Consequently, the solution of the homogeneous equation is

$$y_H(x) = c_1 e^{-2x} + c_2 e^{3x} + c_3 x e^{3x}.$$

6. $\frac{d^2y}{dx^2} + 16y = 0.$

Solution : The characteristic equation is $\lambda^2 + 16 = 0$. The roots of the characteristic equation are $\{\pm 4j\}$. Consequently, the solution of the homogeneous equation is

$$y_H(x) = c_1 \cos 4x + c_2 \sin 4x.$$

7. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0.$

Solution : The characteristic equation is $\lambda^2 - 4\lambda + 13 = (\lambda - 2 - 3j)(\lambda - 2 + 3j)$. The roots of the characteristic equation are $\{2 \pm 3j\}$. Consequently, the solution of the homogeneous equation is

$$y_H(x) = e^{2x}(c_1 \cos 3x + c_2 \sin 3x).$$

8. $\frac{d^7y}{dx^7} + 2\frac{d^5y}{dx^5} + \frac{d^3y}{dx^3} = 0.$

Solution : The characteristic equation is $\left(\lambda^4 + 2\lambda^2 + 1\right)\lambda^3 = \lambda^3\left(\lambda^2 + 1\right)^2$. The roots of the characteristic equation are $\{0, 0, 0, \pm j, \pm j\}$. Consequently, the solution of the homogeneous equation is

$$y_H(x) = c_1 + c_2x + c_3x^2 + (c_4 + c_5x)\cos x + (c_6 + c_7x)\sin x.$$

B. Find the linear, constant coefficient and homogeneous differential equations whose general solutions are given below.

1. $x_H(t) = c_1 + c_2t + c_3e^{3t}\sin t + c_4e^{3t}\cos t + c_5e^{3t}\sin 2t + c_6e^{3t}\cos 2t.$

Solution: Note that $c_1 + c_2t$ corresponds to a double zero root of the characteristic equation. Similarly, $c_3e^{3t}\cos t + c_4e^{3t}\sin t$ and $c_5e^{3t}\cos 2t + c_6e^{3t}\sin 2t$ correspond to two complex roots $3 \pm j$ and $3 \pm 2j$, respecttively. Consequently, the characteristic equation is

$$\lambda^2(\lambda^2 - 6\lambda + 10)(\lambda^2 - 6\lambda + 13) = \lambda^6 - 12\lambda^5 + 59\lambda^4 - 138\lambda^3 + 130\lambda^2.$$

Thus, the differential equation is

$$\frac{d^6x}{dt^6} - 12\frac{d^5x}{dt^5} + 59\frac{d^4x}{dt^4} - 138\frac{d^3x}{dt^3} + 130\frac{d^2x}{dt^2} = 0.$$

2. $x_H(t) = c_1e^t + c_2te^t + c_3e^{2t}\sin t + c_4e^{2t}\cos t + c_5e^{2t}t\sin t + c_6e^{2t}t\cos t.$

Solution : The first two terms correspond to a double root $\lambda = 1$. The last four terms correspond to a double complex root $\lambda = 2 \pm j$. Consequently, the characteristic equation is

$$(\lambda - 1)^2(\lambda^2 - 4\lambda + 5)^2 = \lambda^6 - 10\lambda^5 + 43\lambda^4 - 100\lambda^3 + 131\lambda^2 - 90\lambda + 25.$$

Then, the differential equation is

$$\frac{d^6x}{dt^6} - 10\frac{d^5x}{dt^5} + 43\frac{d^4x}{dt^4} - 100\frac{d^3x}{dt^3} + 131\frac{d^2x}{dt^2} - 90\frac{dx}{dt} + 25x = 0.$$

3. $x_H(t) = c_1e^{2t} + c_2te^{2t} + c_3t^2e^{2t} + c_4e^{-t}\sin 3t + c_5e^{-t}\cos 3t.$

Solution : The first three terms correspond to a triple root $\lambda = 2$. The last two terms correspond to a complex root $\lambda = -1 \pm 3j$. Consequently, the characteristic equation is

$$(\lambda - 2)^3(\lambda^2 + 2\lambda + 10) = \lambda^5 - 6\lambda^4 + 10\lambda^3 - 44\lambda^2 + 104\lambda - 80.$$

Then, the differential equation is

$$\frac{d^5x}{dt^5} - 6\frac{d^4x}{dt^4} + 10\frac{d^3x}{dt^3} - 44\frac{d^2x}{dt^2} + 104\frac{dx}{dt} - 80x = 0.$$

C. Solve the following problems:

1. Given that $\sin x$ is a solution of $\frac{d^4y}{dx^4} + 2\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$. Find the general solution.

Solution : Note that the characteristic equation is $\lambda^4 + 2\lambda^3 + 6\lambda^2 + 2\lambda + 5 = 0$. Since $\sin x$ is a solution, two roots are $\{\pm j\}$. Thus the characteristic equation has $\left(\lambda^2 + 1\right)$ as a factor. Dividing the characteristic equation by $\left(\lambda^2 + 1\right)$, we get

$$\lambda^2 + 2\lambda + 5 = 0.$$

Thus the other roots are $\{-1 \pm 2j\}$. Consequently, the general solution is

$$y(x) = c_1\cos x + c_2\sin x + c_3e^{-x}\cos 2x + c_4e^{-x}\sin 2x.$$

2. Write y_p by using the method of undetermined coefficients for $\frac{d^2y}{dx^2} + y = 3x^2 - 4\sin x$.

Answer :

$$y_p = a + bx + cx^2 + (d + fx)\cos x + (g + hx)\sin x.$$

3. Write y_p by using the method of undetermined coefficients for $\frac{d^4y}{dx^4} - 16y = x^2 \sin 2x + x^4 e^{2x}$.

Answer :

$$y_p = x\left(a + bx + cx^2 + dx^3 + fx^4\right)e^{2x} + x\left(g + hx + kx^2\right)\sin 2x + x\left(m + nx + px^2\right)\sin 2x.$$

D. Find the general solutions of the following differential equations.

1. $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 3xe^x$.

Solution : The standard form of this equation is

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{1}{4}y = \frac{3}{4}xe^x.$$

The characteristic equation is $\lambda^2 + \lambda + \frac{1}{4} = \left(\lambda + \frac{1}{2}\right)^2 = 0$. The roots of the characteristic equation are $\left\{-\frac{1}{2}, -\frac{1}{2}\right\}$. Consequently, the solution of the homogeneous equation is

$$y_H(x) = c_1 e^{-\frac{x}{2}} + c_2 x e^{-\frac{x}{2}}.$$

Since there is no resonance, we calculate xe^x and derivatives. Note that

$$\frac{d}{dx}\left(xe^x\right) = (1 + x)e^x \Rightarrow y_p = (a + bx)e^x.$$

Substituting y_p in the differential equation, we get

$$\begin{aligned} \frac{d^2y_p}{dx^2} + \frac{dy_p}{dx} + \frac{1}{4}y_p &= \frac{3}{4}xe^x \Longrightarrow \\ (a + 2b + bx)e^x + (a + b + bx)e^x + \frac{1}{4}(a + bx)e^x &= \frac{3}{4}xe^x. \end{aligned}$$

This implies that $\frac{9}{4}a + 3b = 0$ and $\frac{9}{4}b = \frac{3}{4} \Longrightarrow b = \frac{1}{3} \Longrightarrow a = -\frac{4}{9}$. Consequently, $y_p = \frac{1}{9}(3x - 4)e^x$ and the general solution of the nonhomogeneous equation is

$$y(x) = c_1 e^{-\frac{x}{2}} + c_2 x e^{-\frac{x}{2}} + \frac{1}{9}(3x - 4)e^x.$$

2. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 3\sin 2x$.

Solution : The characteristic equation is $\lambda^2 + 2\lambda + 5 = 0$. The roots of the characteristic equation are $\{-1 \pm 2j\}$. Consequently, the solution of the homogeneous equation is

$$y_H(x) = c_1 e^{-x}\cos 2x + c_2 e^{-x}\sin 2x.$$

Since there is no resonance $y_p = a\cos 2x + b\sin 2x$. Substituting y_p in the differential equation, we get

$$\begin{aligned} \frac{d^2y_p}{dx^2} + 2\frac{dy_p}{dx} + 5y_p &= 3\sin 2x \Longrightarrow \\ (-4a\cos 2x - 4b\sin 2x) + 2(-2a\sin 2x + 2b\cos 2x) + 5(a\cos 2x + b\sin 2x) &= 3\sin 2x. \end{aligned}$$

This implies that $a + 4b = 0$ and $b - 4a = 3 \Longrightarrow b = \frac{3}{17} \Longrightarrow a = -\frac{12}{17}$. Consequently, $y_p = \frac{1}{17}(3\sin 2x - 12\cos 2x)$, and the general solution of the nonhomogeneous equation is

$$y(x) = c_1 e^{-x}\cos 2x + c_2 e^{-x}\sin 2x + \frac{1}{17}(3\sin 2x - 12\cos 2x).$$

3. $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x = 1 + te^t$.

Solution : The characteristic equation is $\lambda^2 + 2\lambda - 3 = (\lambda + 3)(\lambda - 1) = 0$. The roots of the characteristic equation are $\{-3, 1\}$. Consequently, the solution of the homogeneous equation is

$$x_H(t) = c_1 e^{-3t} + c_2 e^t.$$

For the first term on the right hand side of the equation we have a. For the second term we have $(b + ct)e^t$. Since this term creates resonance, we have

$$\begin{aligned} x_p(t) &= a + \left(bt + ct^2\right) e^t \Longrightarrow \\ x_p'(t) &= \left(b + (b + 2c)t + ct^2\right) e^t, \\ x_p''(t) &= \left(2b + 2c + (b + 4c)t + ct^2\right) e^t. \end{aligned}$$

Substituting these expressions in the equation, we get

$$\left(2b + 2c + (b + 4c)t + ct^2\right) e^t + 2\left(b + (b + 2c)t + ct^2\right) e^t - 3a - 3\left(bt + ct^2\right) e^t = 1 + te^t.$$

This implies that $a = -\frac{1}{3}$, $4b + 2c = 0$, $8c = 1 \Longrightarrow c = \frac{1}{8}$, and $b = -\frac{1}{16}$. Then, we have

$$x_p(t) = -\frac{1}{3} + \frac{1}{16}\left(2t^2 - t\right) e^t$$

which implies that the general solution of the equation is

$$x(t) = c_1 e^{-3t} + c_2 e^t - \frac{1}{3} + \frac{1}{16}\left(2t^2 - t\right) e^t.$$

4. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^x - 10\sin x$.

Solution : The characteristic equation is $\lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$. The roots of the characteristic equation are $\{-1, 3\}$. Consequently, the solution of the homogeneous equation is

$$y_H(x) = c_1 e^{3x} + c_2 e^{-x}.$$

There is no resonance. Therefore, $y_p(x) = ae^x + b\cos x + c\sin x$. Thus, we have

$$\begin{aligned} y_p'(x) &= ae^x - b\sin x + c\cos x, \\ y_p''(x) &= ae^x - b\cos x - c\sin x. \end{aligned}$$

Substituting these expressions in the equation, we get

$$(ae^x - b\cos x - c\sin x) - 2(ae^x - b\sin x + c\cos x) - 3(ae^x + b\cos x + c\sin x) = 2e^x - 10\sin x.$$

Consequently, we get $a = -\frac{1}{2}$, $-4b - 2c = 0$, and $2b - 4c = -10 \Longrightarrow c = 2$ and $b = -1$. Thus, we get

$$y_p(x) = -\frac{1}{2}e^x - \cos x + 2\sin x.$$

Consequently, the general solution of the equation is

$$y(x) = c_1 e^{3x} + c_2 e^{-x} - \frac{1}{2}e^x - \cos x + 2\sin x.$$

5. $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} - 3x = -3te^{-t}$.

Solution : The characteristic equation is $\lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$. The roots of the characteristic equation are $\{-1, 3\}$. Consequently, the solution of the homogeneous equation is

$$x_H(t) = c_1 e^{3t} + c_2 e^{-t}.$$

The derivative of te^{-t} is $(1-t)\,e^{-t}$ and there is a resonance. Hence, $x_p(t) = \left(at + bt^2\right) e^{-t}$. Then, we have

$$\begin{aligned} x_p'(t) &= \left(a + (2b-a)\,t - bt^2\right) e^{-t}, \\ x_p''(t) &= \left(2b - 2a - (4b-a)\,t + bt^2\right) e^{-t}. \end{aligned}$$

Substituting these expressions in the equation, we obtain

$$\left(2b - 2a - (4b-a)\,t + bt^2\right) e^{-t} - 2\left(a + (2b-a)\,t - bt^2\right) e^{-t} - 3\left(at + bt^2\right) e^{-t} = -3te^{-t}.$$

This implies that $2b - 4a = 0$ and $b = \frac{3}{8} \Longrightarrow a = \frac{3}{16}$. Consequently, the general solution of the equation is

$$x(t) = c_1 e^{3t} + c_2 e^{-t} + \frac{3}{16} te^{-t} + \frac{3}{8} t^2 e^{-t}.$$

6. $\frac{d^2x}{dt^2} + 9x = 2\cos 3t + 2\sin 3t$.

   Solution : The characteristic equation is $\lambda^2 + 9 = 0$. The roots of the characteristic equation are $\{\pm 3j\}$. Consequently, the solution of the homogeneous equation is

$$x_H(t) = c_1 \cos 3t + c_2 \sin 3t.$$

   Since there is a resonance $x_p(t) = at\cos 3t + bt\sin 3t$. Then, we get

$$\begin{aligned} x_p'(t) &= a\left(\cos 3t - 3t\sin 3t\right) + b\left(\sin 3t + 3t\cos 3t\right), \\ x_p''(t) &= a\left(-6\sin 3t - 9t\cos 3t\right) + b\left(6\cos 3t - 9t\sin 3t\right). \end{aligned}$$

   Using these expressions in the equation, we obtain

$$a\left(-6\sin 3t - 9t\cos 3t\right) + b\left(6\cos 3t - 9t\sin 3t\right) + 9\left(at\cos 3t + bt\sin 3t\right) = 2\cos 3t + 2\sin 3t.$$

   This implies that $a = -\frac{1}{3}$ and $b = \frac{1}{3}$. Therefore, the general solution is

$$x(t) = c_1 \cos 3t + c_2 \sin 3t - \frac{1}{3} t\cos 3t + \frac{1}{3} t\sin 3t.$$

7. $\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = 3x^2 + 4\sin x - 2\cos x$.

   Solution : The characteristic equation is $\lambda^2\left(\lambda^2 + 1\right) = 0$. The roots of the characteristic equation are $\{0, 0, \pm j\}$. Consequently, the solution of the homogeneous equation is

$$y_H(x) = c_1 + c_2 x + c_3 \cos x + c_4 \sin x.$$

   There is resonance for all the terms on the right hand side of the equation. For the first term on the right $y_{p1} = x^2(a + bx + cx^2)$ because the degree of zero root is two. For the second term $y_{p2} = x\left(d\cos x + f\sin x\right)$ because the multiplicity of the imaginary root is one. Thus, we have

$$\begin{aligned} y_{p1}' &= 2ax + 3bx^2 + 4cx^3, \\ y_{p1}'' &= 2a + 6bx + 12cx^2, \\ y_{p1}''' &= 6b + 24cx, \\ y_{p1}'''' &= 24c. \end{aligned}$$

   Using these expressions in the equation, we get

$$24c + 2a + 6bx + 12cx^2 = 3x^2.$$

This implies that $24c + 2a = 0$, $b = 0$, and $c = \frac{1}{4} \Longrightarrow a = -3$. Consequently, we have $y_{p1} = \frac{1}{4}x^4 - 3x^2$. For the second term, we get

$$\begin{aligned} y'_{p2} &= d(\cos x - x\sin x) + f(\sin x + x\cos x), \\ y''_{p2} &= d(-2\sin x - x\cos x) + f(2\cos x - x\sin x), \\ y'''_{p2} &= d(-3\cos x + x\sin x) + f(-3\sin x - x\cos x), \\ y''''_{p2} &= d(4\sin x + x\cos x) + f(-4\cos x + x\sin x) \end{aligned}$$

Using these expressions in the equation, we get

$$d(4\sin x + x\cos x) + f(-4\cos x + x\sin x) + d(-2\sin x - x\cos x) + f(2\cos x - x\sin x) = 4\sin x - 2\cos x.$$

This implies that $d = 2$, and $f = 1 \Longrightarrow y_{p2} = (2x\cos x + x\sin x)$. Therefore, the general solution is

$$\begin{aligned} y(x) &= c_1 + c_2 x + c_3\cos x + c_4\sin x + y_{p1} + y_{p2}, \\ &= c_1 + c_2 x + c_3\cos x + c_4\sin x + \frac{1}{4}x^4 - 3x^2 + 2x\cos x + x\sin x, \\ &= c_1 + c_2 x - 3x^2 + \frac{1}{4}x^4 + (c_3 + 2x)\cos x + (c_4 + x)\sin x. \end{aligned}$$

8. $\frac{d^2x}{dt^2} + 9x = 9\sec^2 3t, 0 < t < \frac{\pi}{6}$.
Solution : The solution of the homogeneous equation is

$$x_H(t) = c_1\cos 3t + c_2\sin 3t.$$

Let $x_p(t) = c_1(t)\cos 3t + c_2(t)\sin 3t$. Then, using the method of variation of parameters, we get

$$\begin{aligned} \begin{bmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{bmatrix} \begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} &= \begin{bmatrix} 0 \\ 9\sec^2 3t \end{bmatrix} \Longrightarrow \\ \begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} &= \frac{1}{3}\begin{bmatrix} 3\cos 3t & -\sin 3t \\ 3\sin 3t & \cos 3t \end{bmatrix} \begin{bmatrix} 0 \\ 9\sec^2 3t \end{bmatrix} \Longrightarrow \\ \begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} &= \begin{bmatrix} -3\tan 3t\sec 3t \\ 3\sec 3t \end{bmatrix} \Longrightarrow \\ \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix} &= \begin{bmatrix} -\sec 3t + c_{10} \\ \ln|\sec 3t + \tan 3t| + c_{20} \end{bmatrix}. \end{aligned}$$

Therefore, the solution of thedifferential equation is
A: $x(t) = c_{10}\cos 3t + c_{20}\sin 3t + \sin 3t\ln|\sec 3t + \tan 3t| - 1$.

9. $\frac{d^2x}{dt^2} + x = \sec t$.
Solution : The solution of the homogeneous equation is

$$x_H(t) = c_1\cos t + c_2\sin t.$$

Let $x_p(t) = c_1(t)\cos t + c_2(t)\sin t$. Then, using the method of variation of parameters, we get

$$\begin{aligned} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} &= \begin{bmatrix} 0 \\ \sec t \end{bmatrix} \Longrightarrow \\ \begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} &= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 0 \\ \sec t \end{bmatrix} \Longrightarrow \\ \begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} &= \begin{bmatrix} -\tan t \\ 1 \end{bmatrix} \Longrightarrow \\ \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix} &= \begin{bmatrix} \ln|\cos t| + c_{10} \\ t + c_{20} \end{bmatrix}. \end{aligned}$$

Therefore, the solution of the differential equation is
A: $x(t) = c_{10}\cos t + c_{20}\sin t + \cos t\ln|\cos t| + t\sin t$.

10. $\frac{d^2x}{dt^2} + 4x = 3\csc^2 2t,\ 0 < t < \frac{\pi}{2}$.

Solution : The solution of the homogeneous equation is

$$x_H(t) = c_1 \cos 2t + c_2 \sin 2t.$$

Let $x_p(t) = c_1(t)\cos t + c_2(t)\sin t$. Then, using the method of variation of parameters, we get

$$\begin{aligned}
\begin{bmatrix} \cos 2t & \sin 2t \\ -2\sin t & 2\cos t \end{bmatrix} \begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} &= \begin{bmatrix} 0 \\ 3\csc^2 2t \end{bmatrix} \Longrightarrow \\
\begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} 2\cos t & -\sin 2t \\ 2\sin t & \cos 2t \end{bmatrix}\begin{bmatrix} 0 \\ 3\csc^2 2t \end{bmatrix} \Longrightarrow \\
\begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} &= \begin{bmatrix} -\frac{3}{2}\csc 2t \\ \frac{3}{2}\cot 2t \csc 2t \end{bmatrix} \Longrightarrow \\
\begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix} &= \begin{bmatrix} \frac{3}{4}\ln|\csc 2t + \cot 2t| + c_{10} \\ -\frac{3}{4}\csc 2t + c_{20} \end{bmatrix}.
\end{aligned}$$

Therefore, the solution of the differential equation is

A: $x(t) = c_{10}\cos 2t + c_{20}\sin 2t + \frac{3}{4}(\cos 2t \ln|\csc 2t + \cot 2t| - 1)$.

11. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \frac{1}{1+e^x}$.

A: $y(x) = c_1 e^{-x} + c_2 e^{-2x} + \left(e^{-x} + e^{-2x}\right)\ln|1 + e^x|$.

12. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \ln x;\ x > 0$.

A: $y(x) = c_1 e^x + c_2 x e^x - \frac{5}{36}x^3 e^x + \frac{1}{6}x^3 e^x \ln x$.

E. Solve the following initial value problems:

1. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0;\ y(0) = 3, y'(0) = 5$.

Solution : The characteristic equation is $\lambda^2 - \lambda - 12 = 0$. The roots are $\{4, -3\}$. The general solution is

$$y(x) = c_1 e^{4x} + c_2 e^{-3x}.$$

Since $y(0) = 3$, we get $c_1 + c_2 = 3$. Since $y'(x) = 4c_1 e^{4x} - 3c_2 e^{-3x}$ and $y'(0) = 5$, it follows that $4c_1 - 3c_2 = 5$. Consequently, we get $c_1 = 2$ and $c_2 = 1$. Then, the solution of the initial value problem is

$$y(x) = 2e^{4x} + e^{-3x}.$$

2. $9\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + y = 0;\ y(0) = 3, y'(0) = 0$.

Solution : The characteristic equation is $9\lambda^2 - 6\lambda + 1 = 0$. The roots are $\left\{\frac{1}{3}, \frac{1}{3}\right\}$. The general solution is

$$y(x) = c_1 e^{x/3} + c_2 x e^{x/3}.$$

Since $y(0) = 3$, we get $c_1 = 3$. Since $y'(x) = \frac{c_1}{3}e^{x/3} + c_2 e^{x/3} + \frac{x}{3}c_2 e^{x/3}$ and $y'(0) = 0$, it follows that $c_2 = -1$. Then, the solution of the initial value problem is

$$y(x) = 3e^{x/3} - xe^{x/3}.$$

3. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 29y = 0;\ y(0) = 0, y'(0) = 5$.

Solution : The characteristic equation is $\lambda^2 - 4\lambda + 29 = 0$. The roots are $\{2 \pm 5j\}$. The general solution is

$$y(x) = c_1 e^{2x}\cos 5x + c_2 e^{2x}\sin 5x.$$

Since $y(0) = 0$, we get $c_1 = 0$. Since $y'(x) = 2c_2 e^{2x}\sin 5x + 5c_2 e^{2x}\cos 5x$ and $y'(0) = 5$, it follows that $c_2 = 1$. Then, the solution of the initial value problem is

$$y(x) = e^{2x}\sin 5x.$$

4. $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0$; $y(0) = 2, y'(0) = 0, y''(0) = 0$.

Solution : The characteristic equation is

$$\lambda^3 - 2\lambda^2 + 4\lambda - 8 = \lambda^2 (\lambda - 2) + 4 (\lambda - 2) = 0.$$

The roots are $\{2, \pm 2j\}$. The general solution is

$$y(x) = c_1 e^{2x} + c_2 \cos 2x + c_3 \sin 2x.$$

Since $y(0) = 2$, we get $c_1 + c_2 = 2$. Since $y'(x) = 2c_1 e^{2x} - 2c_2 \sin 2x + 2c_3 \cos 2x$ and $y'(0) = 0$, it follows that $2c_1 + 2c_3 = 0$. Furthermore, since $y''(x) = 4c_1 e^{2x} - 4c_2 \cos 2x - 4c_3 \sin 2x$ and $y''(0) = 0$, we also have $4c_1 - 4c_2 = 0$. Therefore, $c_1 = c_2 = 1$ and $c_3 = -1$. Then, the solution of the initial value problem is

$$y(x) = e^{2x} + \cos 2x - \sin 2x.$$

5. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 9x^2 + 4$; $y(0) = 7, y'(0) = -3$.

Solution : The characteristic equation is $\lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0$. The roots are $\{1, 3\}$. The general solution of the homogeneous equation is

$$y_H(x) = c_1 e^x + c_2 e^{3x}.$$

Since there is no resonance $y_p(x) = a + bx + cx^2$. Then, we get $y_p'(x) = b + 2cx \Longrightarrow y_p''(x) = 2c$. Using these expressions in the equation, we obtain

$$2c - 4(b + 2cx) + 3\left(a + bx + cx^2\right) = 9x^2 + 4.$$

This implies that $2c - 4b + 3a = 4$, $c = 3$, and $3b - 8c = 0$. Thus, we get $b = 8$ and $a = 10$. Therefore, the general solution is

$$y(x) = c_1 e^x + c_2 e^{3x} + 10 + 8x + 3x^2.$$

Since $y(0) = 7$, we get $c_1 + c_2 + 10 = 7$. Since $y'(x) = c_1 e^x + 3c_2 e^{3x} + 8 + 6x$ and $y'(0) = -3$, it follows that $c_1 + 3c_2 + 8 = -3$. Therefore, we get $c_2 = -4$ and $c_1 = 1$. Then, the solution of the initial value problem is

$$y(x) = y(x) = e^x - 4e^{3x} + 10 + 8x + 3x^2.$$

6. $\frac{d^2x}{dt^2} + 4x = t^2 + 3e^t$; $x(0) = 0, x'(0) = 2$.

Solution : The solution of the homogeneous equation is

$$x_H(t) = c_1 \cos 2t + c_2 \sin 2t.$$

Since there is no resonance, the particular solution x_p can be written as

$$\begin{aligned} x_p(t) &= \left(a + bt + ct^2\right) + de^t \Longrightarrow x_p'(t) = (b + 2ct) + de^t \Longrightarrow \\ x_p''(t) &= 2c + de^t \Longrightarrow 2c + de^t + 4\left(a + bt + ct^2 + de^t\right) = t^2 + 3e^t. \end{aligned}$$

This implies that $d = \frac{3}{5}$, $c = \frac{1}{4}$, and $b = 0$, $a = -\frac{1}{8}$. Therefore the general solution is

$$x(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t.$$

Since $x(0) = 0$, it follows that

$$x(0) = c_1 - \frac{1}{8} + \frac{3}{5} = 0 \Longrightarrow c_1 = -\frac{19}{40}.$$

Similarly, since $x'(0) = 2$, we get

$$2 = x'(0) = 2c_2 + \frac{3}{5} \Longrightarrow c_2 = \frac{7}{10}.$$

Consequently, the solution of the initial value problem is

A: $x(t) = -\frac{19}{40} \cos 2t + \frac{7}{10} \sin 2t + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t$.

7. $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = te^t + 4$; $x(0) = 1, x'(0) = 1$.
A: $x(t) = -3e^t + 4te^t + \frac{1}{6}t^3e^t + 4$.

8. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 4e^{-x}\cos 2x$; $y(0) = 1, y'(0) = 0$.
A: $y(x) = e^{-x}\cos 2x + \frac{1}{2}e^{-x}\sin 2x + xe^{-x}\sin 2x$.

9. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = x^{-2}e^{-2x}, 0 < x$; $y(1) = \frac{2}{e^2}, y'(1) = -\frac{4}{e^2}$.
A: $y(x) = e^{-2x} + xe^{-2x} - e^{-2x}\ln x$.

10. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{1+x^2}; y(0) = 3, y'(0) = 1$.
Solution : The solution of the homogeneous equation is

$$y_H(x) = c_1e^x + c_2xe^x.$$

Using the method of variation of parameters let $y_p(x) = c_1(x)e^x + c_2(x)xe^x$. Then, we get

$$\begin{aligned}
\begin{bmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{bmatrix}\begin{bmatrix} c_1' \\ c_2' \end{bmatrix} &= \begin{bmatrix} 0 \\ \frac{e^x}{1+x^2} \end{bmatrix} \Longrightarrow \\
\begin{bmatrix} 1 & x \\ 1 & (x+1) \end{bmatrix}\begin{bmatrix} c_1' \\ c_2' \end{bmatrix} &= \begin{bmatrix} 0 \\ \frac{1}{1+x^2} \end{bmatrix} \Longrightarrow \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} (x+1) & -x \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{1+x^2} \end{bmatrix} \Longrightarrow \\
\begin{bmatrix} c_1' \\ c_2' \end{bmatrix} &= \begin{bmatrix} -\frac{x}{1+x^2} \\ \frac{1}{1+x^2} \end{bmatrix} \Longrightarrow \begin{bmatrix} c_1 \\ c_2' \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\ln\left|1+x^2\right| + c_{10} \\ \arctan(x) + c_{20} \end{bmatrix}.
\end{aligned}$$

Consequently, the general solution is

$$y(x) = c_{10}e^x + c_{20}xe^x - \frac{1}{2}\ln\left|1+x^2\right|e^x + xe^x\arctan(x).$$

The fact that $y(0) = 3$ implies that $c_{10} = 3$. Since $y'(0) = 1$ and

$$\begin{aligned}
y'(x) &= 3e^x + c_{20}(x+1)e^x - \frac{xe^x}{1+x^2} - \frac{1}{2}\ln\left|1+x^2\right|e^x + (x+1)e^x\arctan(x) + \frac{xe^x}{1+x^2} \\
&= 3e^x + c_{20}(x+1)e^x - \frac{1}{2}\ln\left|1+x^2\right|e^x + (x+1)e^x\arctan(x),
\end{aligned}$$

it follows that

$$1 = y'(0) = 3 + c_{20} \Longrightarrow c_{20} = -2.$$

Therefore, the solution of the initial value problem is
A: $y(x) = 3e^x - 2xe^x - \frac{1}{2}e^x\ln\left(1+x^2\right) + xe^x\arctan x$.