MATH 216

Homework 1: Solutions

A. Write the type of the following equations:
Ex: $y'' + 2x^3y^2 = 0$ is a second order nonlinear homogeneous ordinary differential equation.

1. $y''' + y(y')^3 = 2x$.

   Answer : It is a third order, nonlinear, and nonhomogeneous ordinary differential equation.

2. $y'' + 2e^{3x}y' + 2y = (x^2+5)^3$.

   Answer : It is a second order, linear, and nonhomogeneous ordinary differential equation.

3. $\frac{dy}{dx} + 3x^2y = 0$.

   Answer : It is a first order, linear, and homogeneous ordinary differential equation.

4. $x''(t) - x^2t^2 = 0$.

   Answer : It is a second order, nonlinear, and homogeneous ordinary differential equation.

B. Show that the following functions are the solutions of the following differential equations:

1. $x'' - 2x' + x = 0; \qquad x(t) = te^t$.

   Solution : Note that

$$\begin{aligned} x' &= (1+t)\,e^t \text{ and } x'' = (2+t)\,e^t \Longrightarrow \\ x'' - 2x' + x &= (2+t)\,e^t - 2\,(1+t)\,e^t + te^t = 0. \end{aligned}$$

2. $y'' + 4y = 0; \qquad y(x) = \cos 2x$

   Solution : y'' can be calculated as follows.

$$\begin{aligned} y' &= -2\sin 2x \text{ and } y'' = -4\cos 2x \Longrightarrow \\ y'' + 4y &= -4\cos 2x + 4\cos 2x = 0. \end{aligned}$$

C. Find the equilibrium points $\left(\frac{dy}{dx} = 0\right)$ of the following differential equations and determine if they are attractive or repulsive (stable or unstable).

1. $\frac{dy}{dx} = -2 - y$.

   Solution : $y = -2$ is the only equilibrium point. If $y > -2$, then $\frac{dy}{dx} < 0$. This implies that y decreases toward -2. If $y < -2$, then $\frac{dy}{dx} > 0$. This implies that y increases toward -2. Thus, $y = -2$ is an attractive (stable) equilibrium point.

2. $\frac{dy}{dx} = 2 - y$.

   Solution : $y = 2$ is the only equilibrium point. If $y > 2$, then $\frac{dy}{dx} < 0$. This implies that y decreases toward 2. If $y < 2$, then $\frac{dy}{dx} > 0$. This implies that y increases toward 2. Thus, $y = 2$ is an attractive (stable) equilibrium point.

3. $\frac{dy}{dx} = (y-2)\,y$.

   Solution : $y = 0$ and $y = 2$ are equilibrium points. If $y < 0$, then $\frac{dy}{dx} > 0$. This implies that y increases toward 0. If $0 < y < 2$, then $\frac{dy}{dx} < 0$. This implies that y decreases toward 0 and away from $y = 2$. Thus, $y = 0$ is an attractive (stable) equilibrium point. If $y > 2$, then $\frac{dy}{dx} > 0$. This implies that y increases and moves away from $y = 2$. This implies that $y = 2$ is a repulsive equilibrium point.

4. $\frac{dy}{dx} = (4-y)\,y$.

Solution : $y=0$ and $y=4$ are equilibrium points. If $y<0$, then $\frac{dy}{dx}<0$. This implies that y decreases and moves away from $y=0$. If $0<y<4$, then $\frac{dy}{dx}>0$. This implies that y increases toward $y=4$ and away from $y=0$. Thus, $y=0$ is a repulsive (unstable) equilibrium point. If $y>4$, then $\frac{dy}{dx}<0$. This implies that y decreases towards $y=4$. This implies that $y=4$ is an attractive (stable) equilibrium point.

5. $\frac{dy}{dx} = \left(y^2-2y-8\right) y$.

Solution : $y=0$, $y=-2$ and $y=4$ are equilibrium points. If $y<-2$, then $\frac{dy}{dx}<0$. This implies that y decreases and moves away from $y=-2$. If $-2<y<0$, then $\frac{dy}{dx}>0$. This implies that y increases toward $y=0$ and away from $y=-2$. Thus, $y=-2$ is a repulsive (unstable) equilibrium point. If $0<y<4$, then $\frac{dy}{dx}<0$. This implies that y decreases towards $y=0$ and moves away from $y=4$. This implies that $y=0$ is an attractive (stable) equilibrium point. If $y>4$, then $\frac{dy}{dx}>0$ and y increases and moves away from $y=4$. This implies that $y=4$ is an repulsive (unstable) equilibrium point.

D. Find the general solutions of the following differential equations:

1. $9yy'+4x=0$.

Solution : This is a separable equation. Thus, we have

$$\begin{aligned} 9y\,dy &= -4x\,dx \Longrightarrow \int 9y\,dy = -\int 4x\,dx + C \Longrightarrow \\ \frac{9}{2}y^2 &= -2x^2 + C \Longrightarrow y = \pm\sqrt{\frac{2}{9}C - \frac{4}{9}x^2} = \pm\frac{2}{3}\sqrt{C_1 - x^2}. \quad \left(C_1 = \frac{C}{2}\right). \end{aligned}$$

2. $y' + (x+1)y^3 = 0$.

Solution : This equation can be written as follows.

$$\begin{aligned} \frac{dy}{y^3} &= -(x+1)\,dx \Longrightarrow \int \frac{dy}{y^3} = -\int (x+1)\,dx + C \Longrightarrow \\ \frac{1}{2y^2} &= \frac{x^2}{2} + x + C \Longrightarrow y = \pm\sqrt{\frac{1}{x^2+2x+2C}}. \end{aligned}$$

3. $\frac{dx}{dt} = 3t(x+1)$.

Solution : This separable equation can be written as follows.

$$\begin{aligned} \frac{dx}{(x+1)} &= 3t\,dt \Longrightarrow \int \frac{dx}{(x+1)} = \int 3t\,dt + C \Longrightarrow \\ \ln|(x+1)| &= \frac{3}{2}t^2 + C \Longrightarrow x(t) = C_1 e^{\frac{3}{2}t^2} - 1. \quad \left(C_1 = e^C\right). \end{aligned}$$

4. $y' + \csc y = 0$.

Solution : This separable equation can be solved as follows.

$$\begin{aligned} \frac{dy}{dx} &= -\frac{1}{\sin y} \Longrightarrow -\sin y\,dy = dx \Longrightarrow -\int \sin y\,dy = \int dx + C \Longrightarrow \\ \cos y &= x + C \Longrightarrow y = \arccos(x+C). \end{aligned}$$

5. $x'\sin 2t = x\cos 2t$.

Solution : This is a separable equation. Therefore, we get

$$\begin{aligned} \frac{dx}{x} &= \cot 2t\,dt \Longrightarrow \ln x = \int \frac{\cos 2t}{\sin 2t}\,dt + C = \frac{1}{2}\ln(\sin 2t) + C \Longrightarrow \\ x &= C_1\sqrt{\sin 2t}. \quad \left(C_1 = e^C\right). \end{aligned}$$

6. $y' + y = 5$.

Solution : Note that e^x is the integrating factor. Thus, we get

$$\begin{aligned} \frac{d}{dx}\left(ye^x\right) &= 5e^x \Longrightarrow ye^x = \int 5e^x\,dx + C = 5e^x + C \Longrightarrow \\ y &= Ce^{-x} + 5. \end{aligned}$$

7. $y' = (y-1)\cot x$.

Solution : Note that this is a separable equation which can be written as follows.

$$\begin{aligned} \frac{dy}{y-1} &= \cot x\,dx \Longrightarrow \int \frac{dy}{y-1} = \int \cot x\,dx + C \Longrightarrow \\ \ln(y-1) &= \ln(\sin x) + C \Longrightarrow y = 1 + C_1 \sin x. \quad \left(C_1 = e^C\right). \end{aligned}$$

8. $\frac{dy}{dx} + \left(\frac{2x+1}{x}\right) y = e^{-2x}$.

Solution : Integrating factor is

$$e^{\int\left(\frac{2x+1}{x}\right)dx} = xe^{2x}.$$

Consequently, we get

$$\begin{aligned} \frac{d}{dx}\left(yxe^{2x}\right) &= xe^{2x}e^{-2x} = x \Longrightarrow yxe^{2x} = \int x\,dx = \frac{x^2}{2} + C \Longrightarrow \\ y &= \frac{x}{2}e^{-2x} + \frac{C}{x}e^{-2x} = \left(\frac{x}{2} + \frac{C}{x}\right)e^{-2x} = \frac{x^2 + C_1}{2xe^{2x}}. \ (C_1 = 2C). \end{aligned}$$

9. $(3x^2 + y^2)dx - 2xydy = 0$.

Solution : Let $M(x,y) = 3x^2 + y^2$ and $N(x,y) = 2xy$. Then, we have

$$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x},$$

which implies that the equation is exact. Thus, it follows that

$$F(x,y) = \int (3x^2 + y^2)\,dx + g(y) = x^3 + xy^2 + g(y).$$

Taking the derivative with respect to y, we obtain

$$\begin{aligned} \frac{\partial F}{\partial y} &= 2xy + g'(y) = N(x,y) = 2xy \Longrightarrow g'(y) = 0 \Longrightarrow \\ g(y) &= C \Longrightarrow F(x,y) = x^3 + xy^2 = C_1, \quad (C_1 = -C). \end{aligned}$$

10. $y' = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$.

Solution : This is a homogeneous equation and we let $v = y/x \Longrightarrow y = vx$. Then, we get

$$\begin{aligned} \frac{dy}{dx} &= v + x\frac{dv}{dx} \Longrightarrow v + x\frac{dv}{dx} = v + \tan(v) \Longrightarrow \frac{dv}{dx} = \frac{\tan(v)}{x} \Longrightarrow \int \frac{dv}{\tan(v)} = \int \frac{dx}{x} + C \Longrightarrow \\ \ln(\sin v) &= \ln x + C \Longrightarrow \sin v = C_1 x \Longrightarrow v = \arcsin(C_1 x) \Longrightarrow y = x\arcsin(C_1 x), \quad \left(C_1 = e^C\right). \end{aligned}$$

11. $e^{\frac{x}{y}}(y-x)\frac{dy}{dx} + y(1 + e^{\frac{x}{y}}) = 0$.

Solution : Let $v = x/y \Longrightarrow y = x/v$. This implies that

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{v} - \frac{x}{v^2}\frac{dv}{dx} \quad \Longrightarrow \frac{1}{v} - \frac{x}{v^2}\frac{dv}{dx} = -\frac{(1+e^v)}{e^v(1-v)} \Longrightarrow \\
\frac{dv}{dx} &= \frac{v^2}{x}\left(\frac{(1+e^v)}{e^v(1-v)} + \frac{1}{v}\right) = \left(\frac{v^2(1+e^v)}{xe^v(1-v)} + \frac{v}{x}\right) \Longrightarrow \\
\frac{e^v(1-v)}{v(v+e^v)}\,dv &= \frac{dx}{x} \Longrightarrow \frac{dv}{v} - \frac{1+e^v}{v+e^v}\,dv = \frac{dx}{x} \Longrightarrow \\
\int \frac{dv}{v} - \int \frac{1+e^v}{v+e^v}\,dv &= \int \frac{dx}{x} + C \Longrightarrow \ln\left(\frac{v}{v+e^v}\right) = \ln x + C \Longrightarrow \\
\frac{v}{v+e^v} &= C_1 x \Longrightarrow \frac{1}{x+ye^{\frac{x}{y}}} = C_1, \;\; \left(C_1 = e^C\right) \Longrightarrow \\
x + ye^{\frac{x}{y}} &= C_2.
\end{aligned}$$

Second way of the solution:
$e^{\frac{x}{y}}(y-x)\frac{dy}{dx} + y(1+e^{\frac{x}{y}}) = 0 \Longrightarrow e^{\frac{x}{y}}(y-x) + y(1+e^{\frac{x}{y}})\frac{dx}{dy} = 0$.Then we use the substitution $v = x/y \Longrightarrow$ $x = vy$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$.Then, we get

$$\begin{aligned}
e^v(y - vy) + y(1+e^v)(v + y\frac{dv}{dy}) &= 0 \\
[e^v(1-v) + v(1+e^v)]dy + (1+e^v)ydv &= 0 \\
(e^v + v)dy &= -(1+e^v)ydv \\
\frac{dy}{y} &= -\frac{(1+e^v)}{e^v+v}dv \\
\int \frac{dy}{y} &= -\int \frac{(1+e^v)}{e^v+v}dv + C \\
\ln y &= -\ln(e^v + v) + C \\
y(e^v + v) &= C_1, \left(C_1 = e^C\right) \\
ye^{\frac{x}{y}} + x &= C_1
\end{aligned}$$

12. $y' = \sqrt{(x+y+1)}$.
Solution : Let $x + y + 1 = v \Longrightarrow y = v - x - 1$. This implies that

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dv}{dx} - 1 = \sqrt{v} \Longrightarrow \frac{dv}{dx} = \sqrt{v} + 1 \Longrightarrow \frac{dv}{\sqrt{v}+1} = dx \Longrightarrow \\
\int \frac{dv}{\sqrt{v}+1} &= \int dx + C = x + C.
\end{aligned}$$

Let $v = z^2$. Then, we have

$$\begin{aligned}
\int \frac{dv}{\sqrt{v}+1} &= \int \frac{2z\,dz}{z+1} = 2\int \frac{u-1}{u}\,du, \text{ where } \; u = z+1 \Longrightarrow \\
2\int \frac{u-1}{u}\,du &= 2u - 2\ln u \Longrightarrow \int \frac{dv}{\sqrt{v}+1} = 2\left(\sqrt{v}+1\right) - 2\ln\left(\sqrt{v}+1\right).
\end{aligned}$$

Consequently, we get

$$\begin{aligned}
2\left(\sqrt{v}+1\right) - 2\ln\left(\sqrt{v}+1\right) &= x + C \Longrightarrow \\
2\left[\sqrt{(x+y+1)} + 1 - 2\ln\left(\sqrt{(x+y+1)}+1\right)\right] - x &= C.
\end{aligned}$$

13. $y' = (x+y+1)^2 - (x+y)$.

Solution : Let $x+y+1 = v \Longrightarrow y = v - x - 1$. This implies that

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dv}{dx} - 1 = v^2 - v + 1 \Longrightarrow \frac{dv}{dx} = v^2 - v + 2 \Longrightarrow \\
\frac{dv}{v^2 - v + 2} &= dx \Longrightarrow \int \frac{dv}{v^2 - v + 2} = \int dx + C \Longrightarrow \\
\int \frac{dv}{\left(v - \frac{1}{2}\right)^2 + \frac{7}{4}} &= \int dx + C = x + C \Longrightarrow \int \frac{4dv}{7\left[\left(\frac{2}{\sqrt{7}}v - \frac{1}{\sqrt{7}}\right)^2 + 1\right]} = x + C.
\end{aligned}$$

Let $u = \frac{2}{\sqrt{7}}v - \frac{1}{\sqrt{7}} \Longrightarrow du = \frac{2}{\sqrt{7}}$. Then, we get

$$\int \frac{4dv}{7\left[\left(\frac{2}{\sqrt{7}}v - \frac{1}{\sqrt{7}}\right)^2 + 1\right]} = \frac{2}{\sqrt{7}} \int \frac{du}{u^2 + 1} = \frac{2}{\sqrt{7}} \arctan u = \frac{2}{\sqrt{7}} \arctan\left(\frac{2}{\sqrt{7}}v - \frac{1}{\sqrt{7}}\right).$$

Consequently, we have

$$\begin{aligned}
\frac{2}{\sqrt{7}} \arctan\left(\frac{2}{\sqrt{7}}v - \frac{1}{\sqrt{7}}\right) &= x + C \Longrightarrow \left(\frac{2}{\sqrt{7}}v - \frac{1}{\sqrt{7}}\right) = \frac{\sqrt{7}}{2} \tan\left(\frac{\sqrt{7}}{2}(x + C)\right) \Longrightarrow \\
\frac{2}{\sqrt{7}}v &= \frac{\sqrt{7}}{2} \tan\left(\frac{\sqrt{7}}{2}(x + C)\right) + \frac{1}{\sqrt{7}} \Longrightarrow v = \frac{7}{4} \tan\left(\frac{\sqrt{7}}{2}(x + C)\right) + \frac{1}{2} \\
y &= \frac{7}{4} \tan\left(\frac{\sqrt{7}}{2}(x + C)\right) - \frac{1}{2} - x.
\end{aligned}$$

14. $(2x + 3y)dx + (3x + 2y)dy = 0$.

Solution : Let $M(x, y) = 2x + 3y$ and $N(x, y) = 3x + 2y$. Then, we have

$$\frac{\partial M}{\partial y} = 3 = \frac{\partial N}{\partial x},$$

which implies that the equation is exact. Thus, it follows that

$$F(x, y) = \int (2x + 3y)\, dx + g(y) = x^2 + 3xy + g(y).$$

Taking the derivative with respect to y, we obtain

$$\begin{aligned}
\frac{\partial F}{\partial y} &= 3x + g'(y) = N(x, y) = 3x + 2y \Longrightarrow g'(y) = 2y \Longrightarrow \\
g(y) &= y^2 + C \Longrightarrow F(x, y) = x^2 + 3xy + y^2 = C_1, \quad (C_1 = -C).
\end{aligned}$$

15. $(x^3 + \frac{y}{x})dx + (y^2 + \ln x)dy = 0$.

Solution : Let $M(x, y) = (x^3 + \frac{y}{x})$ and $N(x, y) = (y^2 + \ln x)$. Then, we have

$$\frac{\partial M}{\partial y} = \frac{1}{x} = \frac{\partial N}{\partial x},$$

which implies that the equation is exact. Thus, it follows that

$$F(x, y) = \int (x^3 + \frac{y}{x})\, dx + g(y) = \frac{x^4}{4} + y \ln x + g(y).$$

Taking the derivative with respect to y, we obtain

$$\begin{aligned}
\frac{\partial F}{\partial y} &= \ln x + g'(y) = N(x,y) = y^2 + \ln x \Longrightarrow g'(y) = y^2 \Longrightarrow \\
g(y) &= \frac{y^3}{3} + C \Longrightarrow F(x,y) = \frac{x^4}{4} + y\ln x + \frac{y^3}{3} = C_1, \quad (C_1 = -C).
\end{aligned}$$

16. $(e^x \sin y + \tan y)dx + (e^x \cos y + x\sec^2 y)dy = 0.$

Solution : Let $M(x,y) = (e^x \sin y + \tan y)$ and $N(x,y) = (e^x \cos y + x\sec^2 y)$. Then, we have

$$\frac{\partial M}{\partial y} = e^x \cos y + \sec^2 y = \frac{\partial N}{\partial x},$$

which implies that the equation is exact. Thus, it follows that

$$F(x,y) = \int (e^x \sin y + \tan y)\, dx + g(y) = e^x \sin y + x\tan y + g(y).$$

Taking the derivative with respect to y, we obtain

$$\begin{aligned}
\frac{\partial F}{\partial y} &= e^x \cos y + x\sec^2 y + g'(y) = N(x,y) = e^x \cos y + x\sec^2 y \Longrightarrow g'(y) = 0 \Longrightarrow \\
g(y) &= C \Longrightarrow F(x,y) = e^x \sin y + x\tan y = C_1, \quad (C_1 = -C).
\end{aligned}$$

17. $(3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0.$

Solution : Let $M(x,y) = (3x^2y + 2xy + y^3)$ and $N(x,y) = (x^2 + y^2)$. Then, we have

$$\frac{\partial M}{\partial y} = 3x^2 + 2x + 3y^2 \neq \frac{\partial N}{\partial x} = 2x$$

Then, we check

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3x^2 + 2x + 3y^2 - 2x}{(x^2 + y^2)} = 3.$$

Consequently, e^{3x} is an integrating factor. Thus, we get

$$M_1(x,y) = e^{3x}(3x^2y + 2xy + y^3) \quad \text{and} \quad N_1(x,y) = e^{3x}(x^2 + y^2)$$

which implies that $M_1(x,y)dx + N_1(x,y)dy = 0$ is exact. Thus, it follows that

$$\begin{aligned}
F(x,y) &= \int e^{3x}(x^2 + y^2)\, dy + g(x) = e^{3x}(x^2y + \frac{y^3}{3}) + g(x) \Longrightarrow \\
\frac{\partial F}{\partial x} &= e^{3x}\left(2xy + 3x^2y + y^3\right) + g'(x) = e^{3x}(3x^2y + 2xy + y^3) \Longrightarrow g(x) = C \Longrightarrow \\
F(x,y) &= e^{3x}\left(x^2y\right) + e^{3x}\frac{y^3}{3} = C.
\end{aligned}$$

18. $ydx + (2x - ye^y)dy = 0.$

Solution : Let $M(x,y) = y$ and $N(x,y) = (2x - ye^y)$. Then, we have

$$\frac{\partial M}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = 2$$

Then, we check

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{-1}{y}.$$

Consequently, y is an integrating factor. Thus, we get

$$M_1(x,y) = y^2 \quad \text{and} \quad N_1(x,y) = (2xy - y^2 e^y)$$

which implies that $M_1(x,y)dx + N_1(x,y)dy = 0$ is exact. Thus, it follows that

$$F(x,y) = \int y^2 \, dx + g(y) = y^2 x + g(y).$$

Taking the derivative with respect to y, we obtain

$$\begin{aligned} \frac{\partial F}{\partial y} &= 2xy + g'(y) = N_1(x,y) = (2xy - y^2 e^y) \Longrightarrow g'(y) = -y^2 e^y \Longrightarrow \\ g(y) &= -y^2 e^y + 2ye^y - 2e^y + C \Longrightarrow F(x,y) = y^2 x - e^y(y^2 - 2y + 2) = C_1, \quad (C_1 = -C). \end{aligned}$$

19. $xy' + y = y^{-2}$

Solution : This equation can be written as follows.

$$y' + \frac{1}{x} y = \frac{1}{x} y^{-2}.$$

Hence, we have a Bernoulli equation with $n = -2$. Let $v = y^3 \Longrightarrow v' = 3y^2 y'$. Thus, we have

$$3y^2 y' + 3y^2 \frac{1}{x} y = 3y^2 \frac{1}{x} y^{-2} \Longrightarrow v' + 3\frac{v}{x} = \frac{3}{x}.$$

The integrating factor is x^3 and we get

$$\frac{d}{dx}\left(x^3 v\right) = 3x^2 \Longrightarrow x^3 v = x^3 + C \Longrightarrow v = 1 + \frac{C}{x^3} \Longrightarrow y = \frac{\left(x^3 + C\right)^{1/3}}{x}.$$

20. $y' = y(xy^3 - 1)$.

Solution : This equation can be written as follows.

$$y' + y = xy^4.$$

Hence, we have a Bernoulli equation with $n = 4$. Let $v = y^{-3} \Longrightarrow v' = -3y^{-4} y'$. Thus, we have

$$-3y^{-4} y' - 3y^{-4} y = -3x \Longrightarrow v' - 3v = -3x.$$

The integrating factor is e^{-3x} and we get

$$\begin{aligned} \frac{d}{dx}\left(e^{-3x} v\right) &= -3xe^{-3x} \Longrightarrow e^{-3x} v = xe^{-3x} + \frac{1}{3} e^{-3x} + C \Longrightarrow v = \frac{3Ce^{3x} + 3x + 1}{3} \\ \Longrightarrow & \quad y = \left(\frac{3}{3Ce^{3x} + 3x + 1}\right)^{\frac{1}{3}}. \end{aligned}$$

21. $(1 + x^2) y' = 2xy(y^3 - 1)$.

Solution : This equation can be written as follows.

$$y' + \frac{2xy}{(1+x^2)} = \frac{2xy^4}{(1+x^2)}.$$

Hence, we have a Bernoulli equation with $n = 4$. Let $v = y^{-3} \Longrightarrow v' = -3y^{-4} y'$. Thus, we have

$$-3y^{-4} y' - \frac{6xy^{-3}}{(1+x^2)} = -\frac{6x}{(1+x^2)} \Longrightarrow v' - \frac{6x}{(1+x^2)} v = -\frac{6x}{(1+x^2)}.$$

The integrating factor is $(1+x^2)^{-3}$ and we get

$$\begin{aligned}
\frac{d}{dx}\left((1+x^2)^{-3}v\right) &= -6x(1+x^2)^{-4} \Longrightarrow (1+x^2)^{-3}v = (1+x^2)^{-3} + C \Longrightarrow v = 1 + C(1+x^2)^3 \\
&\Longrightarrow \quad y = \left(\frac{1}{1+C(1+x^2)^3}\right)^{\frac{1}{3}}.
\end{aligned}$$

22. $xy'' = y'$.

Solution : Let $y' = p$. Then, we get

$$x\frac{dp}{dx} = p \Longrightarrow \frac{dp}{p} = \frac{dx}{x} \Longrightarrow \ln p = \ln x + C \Longrightarrow p = C_1 x. \ \left(C_1 = e^C\right).$$

Since $y' = p$, it follows that

$$y = C_1\frac{x^2}{2} + C_2.$$

23. $x^2y'' + 3xy' = 2$.

Solution : This equation can be written as

$$y'' + \frac{3}{x}y' = \frac{2}{x^2}.$$

Let $y' = p$. Then, we obtain

$$\begin{aligned}
p' + \frac{3}{x}p &= \frac{2}{x^2} \Longrightarrow x^3p' + 3x^2p = 2x \Longrightarrow \frac{d}{dt}\left(x^3p\right) = 2x \Longrightarrow \\
x^3p &= x^2 + C_1 \Longrightarrow p = \frac{1}{x} + \frac{C_1}{x^3} \Longrightarrow y = \ln x - \frac{C_1}{2x^2} + C_2.
\end{aligned}$$

24. $y'' = 2y(y')^3$.

Solution : Let $p = y'$. Then, we get

$$\frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}.$$

In view of the above, this equation can be written as follows.

$$\begin{aligned}
p\frac{dp}{dy} &= 2yp^3 \Longrightarrow \frac{dp}{p^2} = 2y\,dy \Longrightarrow -\frac{1}{p} = y^2 + C_1 \Longrightarrow p = -\frac{1}{y^2+C_1} \Longrightarrow \\
\left(y^2 + C_1\right)\frac{dy}{dx} &= -1 \Longrightarrow \left(y^2 + C_1\right)\,dy = -dx \Longrightarrow \frac{1}{3}y^3 + C_1y = -x + C_2 \Longrightarrow \\
\frac{1}{3}y^3 + C_1y + x &= C_2.
\end{aligned}$$

25. $yy'' + (y')^2 = yy'$.

Solution : Let $p = y'$. Then, we get $\frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$. Then, this equation can be expressed as follows.

$$\begin{aligned}
y\frac{dp}{dy} + p &= y \Longrightarrow \frac{d}{dy}(yp) = y \Longrightarrow yp = \frac{1}{2}y^2 + C_1 \Longrightarrow \frac{dy}{dx} = \frac{1}{2}y + \frac{C_1}{y} \Longrightarrow \\
\frac{dy}{\frac{1}{2}y + \frac{C_1}{y}} &= dx \Longrightarrow \frac{2y\,dy}{y^2 + 2C_1} = dx \Longrightarrow \ln\left(y^2 + 2C_1\right) = x + C_2 \Longrightarrow \\
y &= \pm\sqrt{C_3e^x - 2C_1}, \quad \left(C_3 = e^{C_2}\right).
\end{aligned}$$

E. Solve the following initial value problems:

1. $y' = x^3e^{-y}$; $y(2) = 0$.

Solution : This equation can be written as follows.

$$\frac{dy}{dx} = x^3e^{-y} \Longrightarrow e^y\,dy = x^3\,dx \Longrightarrow e^y = \frac{x^4}{4} + C \Longrightarrow y = \ln\left(\frac{x^4}{4} + C\right).$$

Since $y(2) = 0$, we get

$$0 = y(2) = \ln\left(\frac{2^4}{4} + C\right) \Longrightarrow C = -3 \Longrightarrow y = \ln\left(\frac{x^4}{4} - 3\right)$$

2. $y\frac{dy}{dx} = 4x(y^2+1)^{\frac{1}{2}}$; $y(0) = 1$.

Solution : This equation can be written as follows.

$$\frac{dy}{dx} = \frac{4x(y^2+1)^{\frac{1}{2}}}{y} \Longrightarrow \frac{2y}{(y^2+1)^{\frac{1}{2}}}\,dy = 8x\,dx \Longrightarrow \frac{2}{3}(y^2+1)^{\frac{1}{2}} = 4x^2 + C \Longrightarrow y = \sqrt{(6x^2+C)^2 - 1}.$$

Since $y(0) = 1$, we get

$$1 = y(0) = y = \sqrt{\left(6\,(0)^2 + C\right)^2 - 1} \Longrightarrow C = \sqrt{2} \Longrightarrow y = \sqrt{\left(6x^2 + \sqrt{2}\right)^2 - 1}$$

3. $y' + 3y = 12$; $y(0) = 6$.

Solution : The integrating factor is e^{3x}. Then, we get

$$\begin{aligned} e^{3x}y' + 3ye^{3x} &= 12e^{3x} \Longrightarrow \frac{d}{dx}\left(e^{3x}y\right) = 12e^{3x} \Longrightarrow e^{3x}y = 4e^{3x} + C \\ y &= 4 + Ce^{-3x}. \end{aligned}$$

Since $y(0) = 6$, we get $6 = y(0) = 4 + Ce^{-3(0)} \Longrightarrow C = 2 \Longrightarrow y = 4 + 2e^{-3x}$.

4. $y' = y\cot x$; $y(\frac{\pi}{2}) = 2$.

Solution : This equation can be expressed as follows.

$$\frac{dy}{dx} = y\cot x \Longrightarrow \frac{dy}{y} = \cot x\,dx \Longrightarrow \ln y = \ln(\sin x) + C \Longrightarrow y = C_1\sin x, \quad \left(C_1 = e^C\right).$$

Since $y(\frac{\pi}{2}) = 2$, we get $2 = y(\frac{\pi}{2}) = C_1\sin(\frac{\pi}{2}) \Longrightarrow C_1 = 2 \Longrightarrow y = 2\sin x$.

5. $y' + 3(y-1) = 2x$; $y(0) = 1$.

Solution : This equation can be expressed as follows.

$$\begin{aligned} \frac{dy}{dx} + 3y &= 2x + 3 \Longrightarrow e^{3x}\frac{dy}{dx} + 3ye^{3x} = (2x+3)\,e^{3x} \Longrightarrow \frac{d}{dx}\left(ye^{3x}\right) = (2x+3)\,e^{3x} \Longrightarrow ye^{3x} = \int (2x+3)\,e^{3x}\,dx \\ ye^{3x} &= \frac{(2x+3)}{3}e^{3x} - \frac{2}{3}\int e^{3x}\,dx + C \Longrightarrow ye^{3x} = \frac{2}{3}xe^{3x} + \frac{7}{9}e^{3x} + C \Longrightarrow \\ y &= \frac{1}{9}(6x+7) + Ce^{-3x}. \end{aligned}$$

Since $y(0) = 1$, we get $1 = y(0) = \frac{1}{9}(6(0)+7) + Ce^{-3(0)} \Longrightarrow C = 2/9 \Longrightarrow y = \frac{1}{9}\left(6x + 2e^{-3x} + 7\right)$.

6. $\frac{dy}{dx} = \frac{10}{(x+y)e^{x+y}} - 1$; $y(0) = 0$.

Solution : Let $x + y = v \Longrightarrow y = v - x$. Then, we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{dv}{dx} - 1 = \frac{10}{ve^v} - 1 \Longrightarrow \frac{dv}{dx} = \frac{10}{ve^v} \Longrightarrow \int ve^v\,dv = \int 10\,dx + C \Longrightarrow \\ ve^v - \int e^v\,dv &= 10x + C \Longrightarrow ve^v - e^v = 10x + C \Longrightarrow (x+y-1)e^{x+y} = 10x + C. \end{aligned}$$

Since $y(0) = 0 \Longrightarrow C = -1$. Thus, we get

$$(x+y-1)e^{x+y} = 10x - 1.$$

7. $(4x^2 - 2y^2)y' = 2xy\,;\ y(3) = -5$.

Solution : Dividing both sides by x^2, we get $\left(4 - 2\left(\frac{y}{x}\right)^2\right)\frac{dy}{dx} = 2\frac{y}{x}$. Let $v = y/x \Longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$. Then, we have

$$\begin{aligned} v + x\frac{dv}{dx} &= \frac{2v}{(4-2v^2)} \Longrightarrow x\frac{dv}{dx} = \frac{v}{(2-v^2)} - v = \frac{v^3 - v}{(2-v^2)} \Longrightarrow \\ \frac{dv}{dx} &= \frac{1}{x}\frac{v^3 - v}{(2-v^2)} \Longrightarrow \int \frac{(2-v^2)}{v^3 - v}\,dv = \int \frac{dx}{x} + C. \end{aligned}$$

If we use partial fraction expansion for the first integral, we get

$$\frac{(2-v^2)}{v^3 - v} = \frac{A}{v} + \frac{B}{v-1} + \frac{D}{v+1}.$$

where $A = -2$, $B = 1/2$ and $D = 1/2$. This implies that

$$\begin{aligned} \int \frac{(2-v^2)}{v^3 - v}\,dv &= \int \left(-\frac{2}{v} + \frac{1/2}{v-1} + \frac{1/2}{v+1}\right) = \ln x + C \Longrightarrow \\ \ln\left(\frac{(v^2-1)^{\frac{1}{2}}}{v^2}\right) &= \ln x + C \Longrightarrow \frac{\sqrt{v^2-1}}{v^2} = C_1 x \Longrightarrow \\ \frac{\sqrt{y^2 - x^2}}{y^2} &= C_1,\ \left(C_1 = e^C\right). \end{aligned}$$

Since $y(3) = -5 \Longrightarrow \frac{\sqrt{25-9}}{25} = C_1 \Longrightarrow C_1 = \frac{4}{25}$. Consequently, we get

$$\frac{\sqrt{y^2 - x^2}}{y^2} = \frac{4}{25} \Longrightarrow y^2 - \frac{16}{625}y^4 - x^2 = 0.$$

8. $(x - y)dx + (3x + y)dy = 0\,;\ y(3) = -2$.

Solution : This equation can be written as follows.

$$\frac{dy}{dx} = -\frac{(x-y)}{(3x+y)} = -\frac{(1 - \frac{y}{x})}{(3 + \frac{y}{x})}.$$

Let $v = y/x \Longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$. Then, we get

$$\begin{aligned} v + x\frac{dv}{dx} &= -\frac{(1-v)}{(3+v)} \Longrightarrow \frac{dv}{dx} = -\frac{1}{x}\left(\frac{(1-v)}{(3+v)} + v\right) = -\frac{1}{x}\left(\frac{(v^2+2v+1)}{(3+v)}\right) \Longrightarrow \\ \frac{(3+v)\,dv}{(v+1)^2} &= -\frac{dx}{x} \Longrightarrow \int \frac{A\,dv}{(v+1)} + \int \frac{B\,dv}{(v+1)^2} = -\ln x + C, \end{aligned}$$

where $B=2$ and $A=1$. Consequently, we have

$$\int \frac{dv}{(v+1)} + \int \frac{2\,dv}{(v+1)^2} = -\ln x + C \Longrightarrow \ln(v+1) - \frac{2}{(v+1)} = -\ln x + C.$$

Substituting $v = y/x$, we get

$$\ln(\frac{y+x}{x}) - \frac{2x}{(y+x)} = -\ln x + C \Longrightarrow \ln(y+x) - \frac{2x}{(y+x)} = C.$$

Since $y(3) = -2$, it follows that

$$\ln(-2+3) - \frac{6}{(-2+3)} = C \Longrightarrow C = -6.$$

Consequently, we get

$$\ln(y+x) - \frac{2x}{(y+x)} + 6 = 0.$$

9. $\frac{dy}{dx} = \frac{x^3 - xy^2}{x^2 y}$; $y(1) = 1$.

Solution 1: This equation can be rearranged as follows.

$$\frac{dy}{dx} = \frac{x^3 - xy^2}{x^2 y} = \frac{1 - \left(\frac{y}{x}\right)^2}{\frac{y}{x}}.$$

Let $v = y/x \Longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$. Then, we get

$$\begin{aligned}
v + x\frac{dv}{dx} &= \frac{1-v^2}{v} \Longrightarrow \frac{dv}{dx} = \frac{1}{x}\left(\frac{1-v^2}{v} - v\right) = \frac{1}{x}\left(\frac{1-2v^2}{v}\right) \Longrightarrow \\
\frac{v\,dv}{(1-2v^2)} &= \frac{dx}{x} \Longrightarrow \int \frac{v\,dv}{(1-2v^2)} = \ln x + C \Longrightarrow -\frac{1}{4}\ln\left|1-2v^2\right| = \ln x + C \Longrightarrow \\
\frac{1}{\left|(1-2v^2)\right|^{1/4}} &= e^C x \Longrightarrow \left|(1-2v^2)\right| = \frac{1}{e^{4C}x^4}.
\end{aligned}$$

Since $y(1) = 1$, we get $v(1) = 1$ which implies that $C = 0$. Consequently, we get

$$\left|\left(1 - 2\left(\frac{y}{x}\right)^2\right)\right| = \frac{1}{x^4} \Longrightarrow \left|(x^2 - 2y^2)\right| = \frac{1}{x^2}.$$

Solution 2 : It is an exact equation also. $\frac{dy}{dx} = \frac{x^3 - xy^2}{x^2 y} \Longrightarrow \left(x^3 - xy^2\right)dx - x^2 y dy = 0.$
Let $M = x^3 - xy^2$ and $N = -x^2 y$.Then

$$\frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x}$$

Therefore

$$\begin{aligned}
F(x,y) &= \int (x^3 - xy^2)dx + g(y) = \frac{x^4}{4} - \frac{x^2y^2}{2} + g(y) \Longrightarrow \\
\frac{\partial F}{\partial y} &= -x^2y + g'(y) = -x^2 y \Longrightarrow g'(y) = 0 \\
g(y) &= C \Longrightarrow F(x,y) = \frac{x^4}{4} - \frac{x^2y^2}{2} + C = 0.
\end{aligned}$$

Since $y(1) = 1$, we get $C = \frac{1}{4} \Longrightarrow x^4 - 2x^2y^2 = -1.$

10. $(xy+1)y\ dx + (2y-x)\ dy = 0\,;\ y(0)=3.$

Solution : Let $M = xy^2 + y$ and $N = 2y - x$. Then, we have

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{2xy + 1 - (-1)}{xy^2 + y} = \frac{2}{y}.$$

This implies that the integrating factor is $p(y) = y^{-2}$. Let $M_1 = x + y^{-1}$ and $N_1 = 2y^{-1} - xy^{-2}$.Then, we have

$$\frac{\partial M_1}{\partial y} = -\frac{1}{y^2} = \frac{\partial N_1}{\partial x}$$

which implies that the equation is exact. Thus, we get

$$\begin{aligned} F(x,y) &= \int \left(x + y^{-1}\right)\, dx + g(y) = \frac{x^2}{2} + \frac{x}{y} + g(y) \Longrightarrow \\ \frac{\partial F}{\partial y} &= -\frac{x}{y^2} + g'(y) = 2y^{-1} - xy^{-2} \Longrightarrow g'(y) = 2y^{-1} \Longrightarrow \\ g(y) &= 2\ln y + C \Longrightarrow F(x,y) = \frac{x^2}{2} + \frac{x}{y} + 2\ln y + C = 0. \end{aligned}$$

Since $y(0) = 3$, we get $C = -2\ln 3$. Therefore, it follows that

$$F(x,y) = \frac{x^2}{2} + \frac{x}{y} + 2\ln y = 2\ln 3.$$

11. $y' - \frac{1}{x}y = y^2\,;\ y(1) = 2.$

Solution : This is a Bernoulli equation with $n = 2$. Let $v = y^{1-2} = y^{-1}$. Then, it follows that

$$\frac{dv}{dx} = -y^{-2}\frac{dy}{dx} \Longrightarrow -y^{-2}y' + \frac{1}{x}y^{-1} = -1 \Longrightarrow \frac{dv}{dx} + \frac{v}{x} = -1.$$

Note that the integrating factor is $e^{\int \frac{dx}{x}} = x$. Thus we get

$$\begin{aligned} x\frac{dv}{dx} + v &= -x \Longrightarrow \frac{d}{dx}(xv) = -x \Longrightarrow xv = -\frac{x^2}{2} + C \Longrightarrow v = \frac{C}{x} - \frac{x}{2} \\ &\Longrightarrow\ y = \frac{2x}{2C - x^2}. \end{aligned}$$

Since $y(1) = 2$, we get $C = 1$. Consequently, we have

$$y = \frac{2x}{2 - x^2}.$$

12. $y'' + 4y = 0$; $y(0) = 2$, $y'(0) = -2$.

Solution : Let $p = y'$. Then, we get

$$\begin{aligned} y'' &= \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy} = -4y \Longrightarrow 2p\,dp = -8y\,dy \Longrightarrow p^2 = -4y^2 + C_1 \Longrightarrow \\ p &= -\sqrt{C_1 - 4y^2} \Longrightarrow \frac{dy}{dx} = -\sqrt{C_1 - 4y^2} \Longrightarrow \\ C_1 &= 20,\ \text{because } y'(0) = -2 \text{ and } \frac{dy}{\sqrt{20 - 4y^2}} = -dx \Longrightarrow \\ \int \frac{dy}{\sqrt{5}\sqrt{1 - \left(\frac{y}{\sqrt{5}}\right)^2}} &= -\int 2dx + C_2 \Longrightarrow \arcsin\left(\frac{y}{\sqrt{5}}\right) = -2x + C_2 \Longrightarrow y = \sqrt{5}\sin(-2x + C_2). \end{aligned}$$

Since $y(0) = 2$, we get $\sin(-2(0) + C_2) = \frac{2}{\sqrt{5}}$. This implies that $C_2 = \arcsin\left(\frac{2}{\sqrt{5}}\right)$. Consequently, we obtain.

$$y = \sqrt{5}\left(\frac{2}{\sqrt{5}}\cos(2x) - \sin(2x)\frac{1}{\sqrt{5}}\right) = 2\cos 2x - \sin(2x).$$

13. $xy'' + y' = 4x$; $y(1) = -1$, $y'(1) = 3$.

Solution : Let $p = y'$. Then, we get

$$\begin{aligned} y'' &= \frac{dp}{dx} \Longrightarrow x\frac{dp}{dx} + p = 4x \Longrightarrow \frac{d}{dx}(xp) = 4x \Longrightarrow xp = 2x^2 + C_1 \Longrightarrow \\ \frac{dy}{dx} &= \frac{2x^2 + C_1}{x} \Longrightarrow y = x^2 + C_1 \ln x + C_2. \end{aligned}$$

Since $y'(1) = 3 = \frac{2(1)^2 + C_1}{1} \Longrightarrow C_1 = 1$ and since $y(1) = -1 = (1)^2 + C_1 \ln 1 + C_2 \Longrightarrow C_2 = -2$. Consequently, we get

$$y(x) = x^2 + \ln x - 2.$$