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SIGNATURE:

Question	Points	Score
1	15	
2	10	
3	20	
4	10	
5	20	
6	25	
Total:	100	

- The time limit is 75 minutes.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, mobile phones, smart watches, etc. are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place a box around your answer to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Do not write in the table above.

Elementary Laplace Transforms: Suppose that $a, b \in \mathbb{R}$, $n \in \mathbb{N}$, and $\mathcal{L}\{f(t)\}$ exists and $F(s) = \mathcal{L}\{f(t)\}$

- $\mathcal{L}\{1\} = \frac{1}{s}, s > 0$
- $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \ s > a,$
- $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \ s > 0,$
- $\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$
- $\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$
- $f * g = \int_0^t f(t-\tau)g(\tau)d\tau$
- $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}, s > 0$
- $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}, \ s > 0$
- $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}, \ s > |a|$
- $\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}, \ s > |a|$
- $\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$
- $f * g = g * f = \int_0^t g(t-\tau)f(\tau)d\tau$
- $\mathcal{L}\{f(ct)\} = \frac{1}{c}F(\frac{s}{c}), \ c > 0$
- $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$
- $\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}, \ s > 0$
- $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$
- $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^nF(s)}{ds^n}$
- $\mathcal{L}\{f * g\} = F(s)G(s)$

1. 15 points Find the genaral solution of $y' - \frac{4x}{x^2+1}e^{3y} = 0$.

Solution: The equation given is a seperable differential equation.

$$\frac{dy}{dx} = \frac{4x}{x^2+1}e^{3y}$$
$$e^{-3y}dy = \frac{4x}{x^2+1}dx$$
$$\int e^{-3y}dy = \int \frac{4x}{x^2+1}dx$$
$$-\frac{1}{3}e^{-3y} = 2\ln(x^2+1) + C$$

2. 10 points Find the general solution of $y''' + y' = 0$.

Solution: Let us find the characteristic equation and its roots.

$$r^3 + r = 0 \Rightarrow r(r^2+1) = 0 \Rightarrow r_1 = 0, r_2 = i \text{ and } r_3 = -i$$
$$x(t) = c_1e^{0t} + c_2\cos t + c_3\sin t$$
$$x(t) = c_1 + c_2\cos t + c_3\sin t$$

3. 20 points Find the solution of the initial value problem $y'' - 3y' - 10y = 7e^{-2t} + 10t - 7$, $y(0) = 6$, $y'(0) = 9$.

Solution: Let us find $y_h(t)$, the roots of the characteristic equation are

$$r^2 - 3r - 10 = 0 \Rightarrow (r-5)(r+2) = 0 \Rightarrow r_1 = 5, \quad r_2 = -2$$
$$y_h(t) = c_1 e^{5t} + c_2 e^{-2t}$$

Let us find $y_p(t)$.

$$y_p(t) = Ate^{-2t} + Bt + C \Rightarrow y_p'(t) = Ae^{-2t} - 2Ate^{-2t} + B, \text{ and } y_p''(t) = -4Ae^{-2t} + 4Ate^{-2t}$$
$$y_p'' - 3y_p' - 10y_p = 7e^{-2t} + 10t - 7$$
$$(-4Ae^{-2t} + 4Ate^{-2t}) - 3(Ae^{-2t} - 2Ate^{-2t} + B) - 10(Ate^{-2t} + Bt + C) = 7e^{-2t} + 10t - 7$$
$$-7Ae^{-2t} - 10Bt - 10C - 3B = 7e^{-2t} + 10t - 7 \Rightarrow A = -1, \quad B = -1, \quad C = 1$$
$$y_p(t) = -te^{-2t} - t + 1$$

The generl solution is $y(t) = c_1 e^{5t} + c_2 e^{-2t} - te^{-2t} - t + 1$.

$$y(0) = 6 \Rightarrow y(0) = c_1 + c_2 + 1 = 6$$
$$y'(0) = 9 \Rightarrow y'(t) = 5c_1 e^{5t} - 2c_2 e^{-2t} - e^{-2t} + 2te^{-2t} - 1 \Rightarrow y'(0) = 5c_1 - 2c_2 - 1 - 1 = 9$$
$$\begin{matrix} c_1 + c_2 = 5 \\ 5c_1 - 2c_2 = 11 \end{matrix} \Rightarrow c_1 = 3, \quad c_2 = 2$$
$$y(t) = 3e^{5t} + 2e^{-2t} - te^{-2t} - t + 1$$

4. 10 points Find the inverse Laplace Tranform of $F(s) = \dfrac{8s^2 - s - 16}{(s-1)^2(s+2)}$.

Solution:

$$\frac{8s^2 - s - 16}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$$
$$8s^2 - s - 16 = A(s-1)(s+2) + B(s+2) + C(s-1)^2$$
$$s = 1 \Rightarrow -9 = 3B \Rightarrow B = -3$$
$$s = -2 \Rightarrow 18 = 9C \Rightarrow C = 2$$
$$s = 0 \Rightarrow -16 = -2A + 2B + C \Rightarrow A = 6$$
$$f(t) = \mathcal{L}^{-1}\left\{\frac{8s^2 - s - 16}{(s-1)^2(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{6}{s-1} + \frac{-3}{(s-1)^2} + \frac{2}{s+2}\right\}$$
$$f(t) = 6e^t - 3te^t + 6e^{-2t}$$

5. 20 points Find the solution of the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 2 & -1 & 5 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 17 \\ 2 \\ 5 \end{bmatrix}$$

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Solution: The eigenvalues of the coefficient matrix are $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = 4$. Let us find the corresponding eigenvectors.

$$\lambda_1 = \lambda_2 = 2 \Rightarrow (A - 2I)\mathbf{u} = \mathbf{0} \Rightarrow \begin{bmatrix} 0 & -1 & 5 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \Rightarrow \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$(A - 2I)\mathbf{v} = \mathbf{u} \Rightarrow \begin{bmatrix} 0 & -1 & 5 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \Rightarrow \begin{matrix} 2v_3 = 0 \Rightarrow v_3 = 0 \\ -v_2 + 5v_3 = 1 \Rightarrow v_2 = -1 \\ v_1 \text{ free parameter} \end{matrix} \Rightarrow \mathbf{v} = \begin{bmatrix} v_1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} v_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 4 \Rightarrow (A - 4I)\mathbf{w} = \mathbf{0} \Rightarrow \begin{bmatrix} -2 & -1 & 5 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) e^{2t} + c_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} e^{4t}$$

$$\mathbf{x}(0) = \begin{bmatrix} 17 \\ 2 \\ 5 \end{bmatrix} \Rightarrow \mathbf{x}(0) = \begin{bmatrix} c_1 + 2c_3 \\ c_2 + c_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 17 \\ 2 \\ 5 \end{bmatrix} \Rightarrow \begin{matrix} c_3 = 5 \\ c_2 = -3 \\ c_1 = 7 \end{matrix}$$

$$\mathbf{x}(t) = 7 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} - 3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) e^{2t} + 5 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} e^{4t}$$

$$\mathbf{x}(t) = \begin{bmatrix} 7 - 3t \\ -3 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} 10 \\ 5 \\ 5 \end{bmatrix} e^{4t}$$

6. 25 points Find the general solution of $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$ where $\mathbf{f}(t) = \begin{bmatrix} 2e^{-2x} \\ 6 \end{bmatrix}$, $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$ with the eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$.

Solution: First Way:

$$\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-2t} \\ 6 \end{bmatrix}$$

$$\lambda_1 = -1 \Rightarrow (A+I)\mathbf{v} = \mathbf{0} \Rightarrow \begin{bmatrix} 4 & -2 & 0 \\ 2 & -1 & 0 \end{bmatrix} \Rightarrow \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda_2 = 2 \Rightarrow (A-2I)\mathbf{w} = \mathbf{0} \Rightarrow \begin{bmatrix} 1 & -2 & 0 \\ 2 & -4 & 0 \end{bmatrix} \Rightarrow \mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\mathbf{x}_h(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$$

$$\mathbf{x}_p(t) = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{-2t} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \Rightarrow \mathbf{x}_p'(t) = (-2) \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{-2t} = \begin{bmatrix} -2A_1 \\ -2A_2 \end{bmatrix} e^{-2t}$$

$$\mathbf{x}_p'(t) = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}_p + \begin{bmatrix} 2e^{-2t} \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} -2A_1 \\ -2A_2 \end{bmatrix} e^{-2t} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \left(\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{-2t} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right) + \begin{bmatrix} 2e^{-2t} \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} -2A_1 \\ -2A_2 \end{bmatrix} e^{-2t} = \begin{bmatrix} 3A_1 - 2A_2 + 2 \\ 2A_1 - 2A_2 \end{bmatrix} e^{-2t} + \begin{bmatrix} 3B_1 - 2B_2 \\ 2B_1 - 2B_2 + 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 3B_1 - 2B_2 \\ 2B_1 - 2B_2 + 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -2A_1 \\ -2A_2 \end{bmatrix} = \begin{bmatrix} 3A_1 - 2A_2 + 2 \\ 2A_1 - 2A_2 \end{bmatrix}$$

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

$$\mathbf{x}_p(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t} + \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t} + \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$