Math 216 Summer 2019 **First Exam** **July 4, 2019**

Your Name / Ad - Soyad (**70 min.**) Signature / İmza

Student ID # / Öğrenci No (**mavi tükenmez!**)

Problem	1	2	3	4	5	6	Total
Points:	16	16	16	16	16	20	100
Score:							

Time limit is **75 minutes**. If you need more room on your exam paper, you may use the empty spaces on the paper. You have to show your. Answers without reasonable-even if your results are true- work will either get zero or very little credit.

1. (16 points) Determine the values of r for which the differential equation $6t^2y'' + ty' + y = 0$ has a solution of the form $y = t^r$.

Solution: Since $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Thus DE becomes

$$\begin{aligned} 6t^2y'' + ty' + y = 0 \Rightarrow 6t^2(r(r-1)t^{r-2}) + t(rt^{r-1}) + t^r &= 0 \\ 6r(r-1)t^r + rt^r + t^r &= 0 \\ (6r^2 - 6r + r + 1)t^r &= 0 \\ (3r-1)(2r-1)t^r &= 0 \\ t^r \neq 0 \Rightarrow (3r-1)(2r-1) &= 0 \\ \Rightarrow & \boxed{t \in \{\frac{1}{2}, \frac{1}{3}\}} \end{aligned}$$

2. (16 Points) Solve the initial value problem

$$\frac{dy}{dx} + 2y = 3, \quad y(0) = 1$$

Solution: This is a separable equation. General solution:

$$\begin{aligned} \frac{dy}{3-2y} &= dx \\ -(1/2)\ln|3-2y| &= x + C_0 \\ \ln|3-2y| &= -2x + C_0 \\ 3 - 2y &= Ce^{-2x} \\ y &= (1/2)\left(3 + Ce^{-2x}\right). \end{aligned}$$

Since $y(0) = 1$, $C = -1$. Thus

$$\boxed{y = (1/2)\left(3 - e^{-2x}\right)}$$

3. (16 Points) Solve the initial value problem

$$\frac{dy}{dx} = 4x - y, \quad y(1) = 0$$

Solution: This is a linear equation. Write

$$\frac{dy}{dx} + y = 4x.$$

Multiply by the integrating factor $\mu = e^x$:

$$\begin{aligned} e^x \frac{dy}{dx} + e^x y &= 4xe^x; \\ \frac{d}{dx}(e^x y) &= 4xe^x; \\ e^x y = \int 4xe^x \,\mathrm{d}x &= 4xe^x - 4\int e^x \,\mathrm{d}x \\ &= (4x-4)e^x + C \end{aligned}$$

where we used integration by parts. Since $y(1) = 0$, and

$$e^x y = (4x-4)e^x.$$

Thus

$$y = 4x - 4.$$

4. (16 Points) Find an implicit solution to the differential equation

$$(e^{xy}y + 3x^2y - y^2)dx + (e^{xy}x + x^3 - 2xy - 1)dy = 0$$

that satisfies $y(0) = 2$.

Solution: Let

$$M = e^{xy}y + 3x^2y - y^2, \text{ and } N = e^{xy}x + x^3 - 2xy - 1.$$

Then

$$M_y = xye^{xy} + e^{xy} + 3x^2 - 2y, \text{ and } N_x = xye^{xy} + e^{xy} + 3x^2 - 2y.$$

Since $M_y = N_x$, the equation is exact. The general solution is of the form $F(x,y) = C$, where

$$\begin{cases} \dfrac{\partial F}{\partial x} = e^{xy}y + 3x^2y - y^2 \\ \\ \dfrac{\partial F}{\partial y} = e^{xy}x + x^3 - 2xy - 1. \end{cases}$$

The first equation implies that $F(x,y)$ is of the form

$$F(x,y) = \int (e^{xy}y + 3x^2y - y^2)\partial x = e^{xy} + x^3y - xy^2 + g(y)$$

the second equation implies that

$$g'(y) = -1$$

so $g(y) = -y$ and $F(x,y) = e^{xy} + x^3y - xy^2 + -y$. Thus the answer to the problem is

$$e^{xy} + x^3y - xy^2 - y = C$$

Using $y(1) = 0$, $C = 1$. Hence the solution to initial value problem is $e^{xy} + x^3y - xy^2 - y = 1$

5. (16 Points) Find all solutions of the equation

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}.$$

Solution:

This is a homogeneous equation which can be written as

$$\frac{dy}{dx} = \frac{(y/x) - 4}{1 - (y/x)}.$$

Change of variables leads to $v = y/x$, $y = xv$, $dy = xdv + vdx$ leads to

$$x\frac{dv}{dx} + v = \frac{v - 4}{1 - v}$$

which is a separable equation. The general solution:

$$x\frac{dv}{dx} = \frac{v-4}{1-v} - v = \frac{v^2 - 4}{1 - v};$$

$$\frac{1-v}{v^2-4}dv = \frac{dx}{x};$$

$$\int \frac{1-v}{v^2-4}dv = \int \frac{dx}{x}$$

Partial fractions gives

$$\frac{-v+1}{v^2-4} = \frac{A}{(v-2)} + \frac{B}{v+2};$$

$$-v+1 = A(v+2) + B(v-2) \Rightarrow A = -1/4, B = -3/4;$$

$$\int \frac{-1/4}{v-2}dv + \int \frac{-3/4}{v+2}dv = \int \frac{dx}{x};$$

$$\ln(|v-2|^{-1/4}|v+2|^{-3/4}) = \ln|x| + C;$$

$$\frac{1}{(v-2)(v+2)^3} = Cx^4;$$

$$x^4(v-2)(v+2)^3 = C.$$

Note that the last formula includes the singular solutions $v = 2$ and $v = -2$. Substituting $y = vx$ gives

$$\boxed{(y-2x)(y+2x)^3 = C}$$

6. Given the equation

$$y' = y^2(y^2-1).$$

(a) (10 Points) Solve this equation as separable equation.

Solution: Separating variables we obtain $\frac{dy}{y^2(y^2-1)} = dx$. By partial fractions,

$$\begin{aligned} \left(\frac{0}{y} - \frac{1}{y^2} + \frac{1/2}{y-1} - \frac{1/2}{y+1}\right) \mathrm{d}y &= \mathrm{d}x \\ -\int \frac{1}{y^2}\,\mathrm{d}y + \frac{1}{2}\int \frac{1}{y-1}\,\mathrm{d}y - \frac{1}{2}\int \frac{1}{y+1}\,\mathrm{d}y &= \int \mathrm{d}x \\ 2 + \frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y+1| &= x + C_0 \\ y\ln\left|\frac{y-1}{y+1}\right| &= x - 2 + Cy \end{aligned}$$

(b) (10 Points) Determine the critical (equilibrium) points, and classify each one as stable, unstable or semistable. (A critical point is semistable if it is stable on one side and is unstable on the other side.)

Solution: Solving $f(y) = y^2(y^2-1) = 0$, we have $y = 1$, 0 or 1. If $y \in (-\infty, -1)$, then $y'(t) = f(y) > 0$, hence $y(t)$ is increasing in t. If $y \in (-1, 0)$, then $y'(t) = f(y) < 0$, hence $y(t)$ is decreasing in t. If $y \in (0, 1)$, then $y'(t) = f(y) < 0$, hence $y(t)$ is decreasing in t. If $y \in (1, \infty)$, then $y'(t) = f(y) > 0$, hence $y(t)$ is increasing in t. Therefore, $y = 1$ is a stable critical point, $y = 0$ is a semi-stable critical point and $y = 1$ is an unstable critical point.