Your Name / Ad - Soyad (**70 min.**) Signature / İmza

Student ID # / Öğrenci No (**mavi tükenmez!**)

Problem	1	2	3	4	5	Total
Points:	16	18	20	30	16	100
Score:						

Time limit is **70 minutes**. If you need more room on your exam paper, you may use the empty spaces on the paper. You have to show your. Answers without reasonable-even if your results are true- work will either get zero or very little credit.

1. (16 points) Solve the initial value problem

$$y'' - 6y' + 9y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution: From the auxiliary equation $m^2 - 6m+) = 0$, we have $(m-3)^2 = 0$ and we get $m_1 = m_2 = 3$ as the roots so that the corresponding general solution

$$y = c_1 e^{3x} + c_2 x e^{3x}.$$

Imposing the condition $y(0) = 1$, we see from $c_1 e^0 + c_2 0 e^0 = 1$ that $c_1 = 1$. Differentiating $y' = 3c_1 e^{3x} + c_2 e^{3x} + 3c_2 x e^{3x}$ and then using $y'(0) = 0$ gives $y' = 3c_1 e^0 + c_2 e^0 + 3c_2 0 e^0 = 0$ and so $3c_1 + c_2 = 0$. Hence $c_2 = -3$. Hence the solution of the IVP is

$$y = e^{3x} - 3xe^{3x} = \boxed{e^{3x}(1-3x)}.$$

2. (18 Points) Find the general solution.

$$y'' + 2y' - 3y = e^{-3x} + 3x$$

Solution: The complementary function is $y_c = c_1 e^x + c_2 e^{-3x}$ and we assume $y_p = Axe^{-3x} + Bx + C$. Then
$y_p' = Ae^{-3x} - 3Axe^{-3x} + B$ and
$y_p'' = -3Ae^{-3x} - 3Ae^{-3x} + 9Axe^{-3x} = -6Ae^{-3x} + 9Axe^{-3x}$. Substituting into the differential equation we get

$$\begin{aligned} -6Ae^{-3x} + 9Axe^{-3x} + 2(Ae^{-3x} - 3Axe^{-3x} + B) - 3(Axe^{-3x} + Bx + C) &= e^{-3x} + 3x \\ -4Ae^{-3x} + (-3B)x + 2B - 3C &= e^{-3x} + 3x \\ -4A = 1, \ -3B = 3, \text{ and } \ 2B - 3C &= 0 \\ A = -1/4, \ B = -1, \ C &= -2/3 \end{aligned}$$

Hence $y_p = -\frac{1}{4}xe^{-3x} - x - \frac{2}{3}$. Hence the general solution is

$$\boxed{y = y_c + y_p = c_1 e^x + c_2 e^{-2x} - \frac{1}{4}xe^{-3x} - x - \frac{2}{3}}$$

3. (20 Points) Use variation of parameters to find the general solution to

$$y'' + 4y = \sec(2x),$$

(if $y_c = c_1\cos(2x) + c_2\sin(2x)$).

Solution: From the auxiliary equation $m^2+4=0$ we have $y_c = c_1\cos(2x)+c_2\sin(2x)$. With the identifications $y_1=\cos(2x)$ and $y_2=\sin(2x)$, we next compute the Wronskian:

$$W(\cos(2x),\sin(2x)) = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix} = 2\cos^2(2x)+2\sin^2(2x) = 2 \neq 0$$

Since the given differential equation is already in standard form (that is, the coefficient of y'' is 1), we identify $f(x)=\sec(2x)$. We now obtain

$$W_1 = \begin{vmatrix} 0 & \sin(2x) \\ \sec(2x) & 2\cos(2x) \end{vmatrix} = 0 - \sec(2x)\sin(2x) = -\frac{\sin(2x)}{\cos(2x)} = -\tan(2x)$$

and

$$W_2 = \begin{vmatrix} \cos(2x) & 0 \\ -2\sin(2x) & \sec(2x) \end{vmatrix} = 1$$

and so

$$u_1' = \frac{W_1}{W} = \frac{-\tan(2x)}{2} = -\frac{1}{2}\tan(2x) \quad u_2' = \frac{W_2}{W} = \frac{1}{2}$$

Integrating we have

$$u_1 = \int (-\frac{1}{2}\tan(2x))\,\mathrm{d}x = \frac{1}{4}\ln|\cos(2x)| \quad \text{and} \quad u_2 = \int \frac{1}{2}\,\mathrm{d}x = \frac{1}{2}x$$

Therefore, the particular solution is given by

$$y_p = u_1\cos(2x) + u_2\sin(2x) = \frac{1}{4}\ln|\cos(2x)|\cos(2x) + \frac{1}{2}x\sin(2x)$$

Then the general solution is

$$y_g = y_c + y_p = \boxed{c_1\cos(2x) + c_2\sin(2x) + \frac{1}{4}\ln|\cos(2x)|\cos(2x) + \frac{1}{2}x\sin(2x)}$$

4. Compute the following.

(a) (10 Points) $\mathscr{L}\left\{e^{2t}(t-1)^2\right\}$

Solution: We first expand the square and then multiply.

$$\mathscr{L}\left\{e^{2t}(t-1)^2\right\} = \mathscr{L}\left\{e^{2t}(t^2-2t+1)\right\} = \mathscr{L}\left\{t^2e^{2t}-2te^{2t}+e^{2t}\right\}$$
$$= \boxed{\frac{2}{(s-2)^3}-\frac{2}{(s-2)^2}+\frac{1}{s-2}}$$

(b) (10 Points) $\mathscr{L}\left\{f(t)\right\}$ where $f(t) = \begin{cases} 1-t, & 0<t<1 \\ 0, & t>1 \end{cases}$

Solution: By integrating by parts, we have

$$\mathscr{L}\left\{f(t)\right\} = \int_0^\infty e^{-st}f(t)\,\mathrm{d}t = \int_0^1 (1-t)e^{-st}\,\mathrm{d}t = \left(-\frac{1}{s}(1-t)e^{-st}+\frac{1}{s^2}e^{-st}\right)\Bigg|_0^1$$
$$= \boxed{\frac{1}{s^2}e^{-s}+\frac{1}{s}-\frac{1}{s^2}, \quad s>0}$$

(c) (10 Points) $\mathscr{L}^{-1}\left\{\frac{5s}{(s-2)^2}\right\}$

Solution:

$$\mathscr{L}^{-1}\left\{\frac{5s}{(s-2)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{5(s-2)+10}{(s-2)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{5}{s-2}+\frac{10}{(s-2)^2}\right\}$$
$$= \boxed{5e^{2t}+10te^{2t}}$$

5. (16 Points) Solve, using Laplace transforms, the initial value problem.

$$y'' - 5y' - 6y = 2e^t, \quad y(0) = 0, \quad y'(0) = 1$$

Although it is possible to solve this with other methods, you will receive credit only if you use Laplace transforms.

Solution: The Laplace transform of the equatin is

$$\mathscr{L}\{y'' - 5y' - 6y\} = \mathscr{L}\{2e^t\} \Rightarrow \mathscr{L}\{y''\} - 5\mathscr{L}\{y'\} - 6\mathscr{L}\{y\} = \mathscr{L}\{2e^t\}$$

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) - 5[s\mathscr{L}\{y\} - y(0)] - 6\mathscr{L}\{y\} = \frac{2}{s-1}$$

$$Y(s)(s^2 - 5s - 6) = \frac{2}{s-1} + 1 \Rightarrow Y(s) = \frac{s+1}{(s-1)(s-6)(s+1)}$$

$$Y(s) = \frac{1}{(s-1)(s-6)}$$

$$Y(s) = -\frac{1}{5}\frac{1}{s-1} + \frac{1}{5}\frac{1}{s-6}$$

$$\boxed{y(t) = -\frac{1}{5}e^t + \frac{1}{5}e^{6t}}$$

$f(t)$	$F(s) = \mathscr{L}[f](s)$	
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$t^n \quad (n \in \mathbb{N})$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > \|a\|$
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > \|a\|$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$t^n e^{at} \quad (n \in \mathbb{N})$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$e^{ct}f(t)$	$F(s-c)$	
$f(ct) \quad (c > 0)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$	
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	